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FURTHER RESULTS ON THE ANALYSIS OF THE VARIATION OF LATITUDE

A. M. Walker and Andrew Young

(Received 1957 January 1)

Summary

In this paper we give further results of our analysis of the observations of the variation of latitude. We find that the best results are those derived from the unsmoothed series of values given at monthly intervals by the International Service.

There is evidence of inhomogeneity in the data because of which it is difficult to obtain a reliable estimate of the damping factor. We discuss the bias introduced into the estimates of the period and damping by the inhomogeneity of the data and by other causes.

1. In our previous paper (**1**) we formulated a statistical model of the variation of latitude and gave a detailed analysis of its properties. We obtained a computational procedure which we applied to a 35-year series of observations given by the International Service. These particular observations were examined because they were unsmoothed and we hoped that the effect of smoothing might be seen when the results were compared with those obtained by Jeffreys (**2**) who used smoothed observations. However, the two sets of observations differed in another important respect—the z-term had not been taken into account in the set which we analysed but had been in the set analysed by Jeffreys.

In this paper we give much more detailed results and examine some of the points raised in the previous paper.

2. Through the good offices of Professor T. Nicolini of Naples we have obtained much more extensive sets of observations, one set of which forms series of unsmoothed results covering the interval from 1900 to 1955 at monthly intervals, the other forming series of smoothed results for the interval from 1890 to 1955 at 1/10 yearly intervals. In both sets the z-term has been removed. These sets are collated from several sources and, as some of the results are as yet unpublished in easily accessible form, they are given in Tables I *a* and I *b*. The first series (Table I *a*) are exhibited graphically in Fig. 1 (*a*, *c*).

Observations of the variation of latitude were made at a number of uncoordinated observatories prior to the inception of the International Service. The results were collated by Albrecht and form the basis for the values given in Table I *b* for the interval 1890·0 to 1899·8. The international programme started at the end of 1899. Until 1922·7, the work was under the direction firstly of Albrecht and then Wanach and Mahnkopf at Potsdam. From 1922·7 to 1934·9 the work was directed by Kimura at Mizusawa; he was succeeded in 1935 by Professor Carnera and in 1948 by Professor Cecchini of Turin, the present director. The methods of reduction used by the different directors have not been the same and according to Melchior (**3**, p. 38) the results require

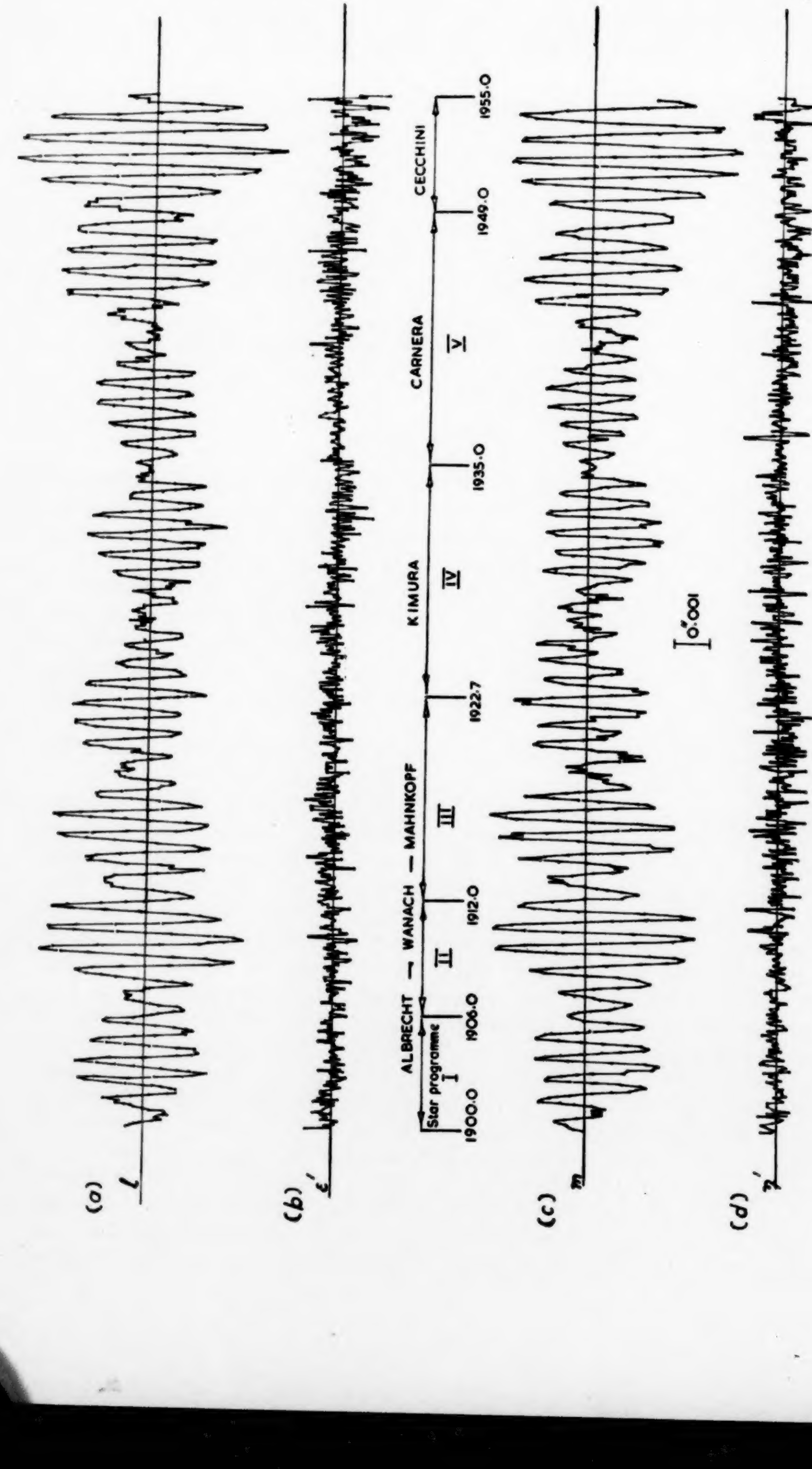

FIG. 1.—*Variation of latitude. Motion* (l, m) *at monthly intervals and residual motion* (ϵ', η').

corrections; the published results can, he states, be brought into agreement by applying the following corrections*

$$\begin{aligned} x_w &= x_k + 0''\cdot048 = x_c + 0''\cdot040 = x_{ce} + 0''\cdot004, \\ y_w &= y_k + 0''\cdot072 = y_c + 0''\cdot060 = y_{ce} - 0''\cdot002, \end{aligned} \tag{1}$$

where (x,y) are the coordinates of the pole† as published and w, k, c, ce refer to Wanach, Kimura, Carnera and Cecchini respectively. As the values of x and y seldom exceed $\pm 0''\cdot300$, these corrections seem to be large.

3. For the international programme, observations are made at several observatories on the parallel of latitude 39° 8′ N. Ideally there should be a large number of stations distributed symmetrically on the parallel. However, at most, six stations have been in use, and for long periods only three have functioned. The stations with details of the intervals during which they functioned are given in Table II.

If $\Delta\phi_i$ is the value of the variation of latitude of the station, i, measured from its mean latitude, then (x,y) are given by

$$x \cos \lambda_i + y \sin \lambda_i = \Delta\phi_i - z \tag{2}$$

where λ_i is the longitude of the station and z is Kimura's z-term which arises from a number of causes which are not all perfectly understood. These causes include errors in the adopted values of the proper motions and declinations of the groups of stars observed, and also perhaps local effects such as refraction and oscillations of the vertical.

When more than three stations were included, equations (2) were solved by least squares. For example, from 1901·7 to 1906·0 the values of x were found from the formula

$$x = -0\cdot437\Delta\phi_1 + 0\cdot139\Delta\phi_2 + 0\cdot426\Delta\phi_3 + 0\cdot101\Delta\phi_4 + 0\cdot042\Delta\phi_5 - 0\cdot272\Delta\phi_6,$$

whereas, when only three stations were used from 1922·7 to 1935·0, the formula became

$$x = -0\cdot396\Delta\phi_1 + 0\cdot591\Delta\phi_3 - 0\cdot195\Delta\phi_6.$$

There are evidently quite considerable changes in the relative weights of the contributions of different stations.

Because of precession and proper motion, no individual star can be used in the observing programme indefinitely and the programme has been changed on a number of occasions. From 1899·9 to 1905·9 each of the twelve groups of stars included eight pairs, of which two were "refraction-pairs" of larger declination than the others. At 1906·0 these were replaced by normal pairs, and, in effect, the number of stars used was thus increased from 144 to 192, of which 132 were in the original programme. Further changes were made at 1912·0, 1922·7 and 1935·0. At each change the number of new stars introduced was not great. In fact, in the programme used since 1955·0, there are still ten of the original pairs of stars used since 1899 (see Cecchini, **4**). From conversations which one of us (A. Y.) has had with Professors Cecchini and

* If these suggested corrections are valid they should agree with the means of c, c' given by Jeffreys (**2**, p. 146.) The latter however average to $0''\cdot01$ at most over the intervals mentioned.

† In (1) (x, y) are the coordinates of the pole referred to a left-handed set of axes. We follow Jeffreys, and use a right-handed set with $l = x$ and $m = -y$. We use (x, y) to denote the free component of the motion after the systematic, annual component (l_s, m_s) has been deducted from (l, m)—cf. **1**, equation (6).

TABLE I*a* *Results of Variations of Latitude :*

l

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sep.	Oct.	Nov.	Dec.
1899												−7
1900	69	49	40	15	−15	−46	−32	−44	−70	−64	−65	−8
1	8	44	53	68	102	94	143	91	47	−10	−71	−99
2	−102	−76	−48	−15	71	128	203	216	167	53	−34	−119
3	−162	−199	−136	−103	−46	37	95	165	226	175	116	−3
4	−104	−153	−172	−163	−161	−102	−14	67	133	183	153	117
1905	30	−30	−130	−122	−146	−134	−90	−27	61	115	144	132
6	44	12	−20	−42	−94	−117	−110	−96	−36	−9	20	40
7	66	71	52	67	66	10	1	−11	−57	−147	−122	−77
8	−35	35	99	153	205	223	232	177	79	−71	−209	−246
9	−276	−223	−193	−85	43	189	280	347	309	163	−45	−166
1910	−205	−253	−307	−277	−216	−78	96	244	324	322	230	95
1	−47	−145	−197	−237	−207	−172	−126	25	144	249	314	263
2	148	73	9	−72	−114	−154	−116	−96	−64	−16	82	116
3	116	143	138	136	136	123	108	8	−26	−106	−83	−72
4	−84	−36	74	144	196	182	198	168	92	8	−115	−158
1915	−196	−158	−150	−34	24	107	205	270	289	183	71	−86
6	−144	−168	−191	−186	−120	−2	80	191	289	310	222	116
7	−5	−38	−112	−172	−126	−70	−30	59	160	174	172	160
8	88	−9	−28	−43	−66	−76	−74	12	58	82	99	100
9	112	68	52	92	80	86	70	50	−9	−58	−68	−42
1920	21	59	28	126	152	198	212	203	170	82	22	−72
1	−76	−102	−62	12	65	176	214	244	236	146	72	22
2	−82	−104	−90	−38	2	92	172	254	210	167	95	−26
3	−89	−125	−178	−127	−90	−11	79	176	207	198	160	72
4	−9	−62	−111	−96	−97	−78	−9	84	94	109	94	71
1925	36	−41	−47	−70	−24	−24	−17	54	58	76	61	−2
6	16	−75	−99	−95	−95	−61	−8	16	54	52	16	28
7	−5	71	−14	39	21	24	36	59	37	19	15	−29
8	−75	−70	−68	−22	−13	−12	6	56	−11	−59	−53	−73
9	−58	−116	−44	33	46	101	110	117	75	−9	−59	−97
1930	−146	−148	−88	−50	5	79	154	173	127	60	11	−121
1	−151	−148	−172	−95	−41	70	165	193	206	72	25	−56
2	−168	−238	−186	−111	−119	−69	15	137	187	159	100	36
3	−85	−120	−138	−162	−117	−91	−28	68	73	74	55	27
4	−26	−72	−130	−167	−123	−74	−11	37	28	51	20	37
1935	−2	9	11	8	11	23	50	42	10	−22	−46	−60
6	−67	−62	−44	−8	45	92	102	89	50	7	−33	−81
7	−129	−133	−80	−12	33	89	128	139	75	30	−22	−95
8	−134	−125	−102	−66	−22	37	129	176	172	100	36	−76
9	−104	−115	−112	−48	−14	36	100	172	179	163	105	22
1940	−42	−92	−122	−133	−84	8	65	99	121	120	58	22
1	55	15	3	−31	7	16	19	57	67	79	93	23
2	16	−7	1	−7	12	18	3	−24	−12	13	0	22
3	43	72	107	98	147	132	98	120	38	−23	7	−67
4	−33	−7	10	76	191	278	274	279	151	−19	−55	−152
1945	−153	−174	−137	−66	92	171	234	279	305	239	100	74
6	−115	−176	−192	−135	−82	−39	114	166	270	283	244	208
7	73	−33	−106	−183	−194	−129	−20	73	148	226	254	253
8	245	198	164	79	−18	−75	−58	−74	−112	−48	−14	120
9	124	234	240	238	205	204	115	−5	−101	−144	−178	−200
1950	−138	−67	76	169	266	352	370	307	132	−18	−185	−281
1	−324	−328	−199	−55	139	259	397	454	377	259	95	−112
2	−308	−369	−420	−247	−158	45	216	364	428	415	289	66
3	−80	−199	−336	−353	−351	−192	−42	120	280	336	298	126
4	−12	−93	−186	−222	−269	−144	−80	27	34	70	88	4

Sources:
1899 Dec.–1905 Dec. Albrecht and Wanach
1906 Jan. –1911 Dec. Wanach
1912 Jan. –1922 July Wanach and Mahnkopf
1922 Aug.–1934 Dec. Kimura
1935 Jan. –1940 Dec. Carnera
1941 Jan. –1948 Dec. Carnera

1949 Jan. –1954 Dec. Cecchini

(*Monthly interval*) : *unit* 0″·001.

Jan.	Feb.	Mar.	Apr.	May	June	*m* Jul.	Aug.	Sep.	Oct.	Nov.	Dec.	
											-61	1899
-4	33	35	54	63	80	86	63	19	-29	-58	-26	1900
-61	-26	-67	-41	-25	33	73	143	130	113	82	40	1
-30	-94	-147	-169	-197	-184	-122	-42	37	125	115	71	2
17	-34	-126	-137	-220	-234	-238	-197	-91	55	149	146	3
116	66	7	-56	-116	-189	-209	-193	-142	-61	6	79	4
147	150	122	102	25	-21	-117	-178	-192	-162	-83	-3	1905
40	51	55	31	23	-22	-76	-101	-129	-131	-130	-147	6
-94	-47	-33	1	29	56	77	79	33	-26	-113	-177	7
-211	-208	-206	-167	-112	-9	40	132	170	167	120	23	8
-73	-179	-257	-298	-319	-290	-210	-61	95	218	284	266	9
190	74	-46	-157	-275	-320	-349	-314	-163	68	192	286	1910
280	235	130	32	-92	-197	-311	-346	-311	-190	21	76	1
156	210	186	189	104	54	1	-124	-134	-166	-170	-117	2
-70	-44	-4	54	64	114	66	102	109	45	-32	-89	3
-122	-98	-137	-87	-87	-75	-12	79	136	210	129	47	4
14	-116	-198	-270	-284	-224	-210	-92	18	204	268	196	1915
231	124	-19	-108	-152	-276	-282	-184	-30	82	222	251	6
298	253	188	84	-6	-98	-220	-201	-171	-42	24	98	7
190	118	61	24	-23	-94	-139	-126	-122	-66	-57	-68	8
-53	-9	-82	-23	50	4	14	57	14	-10	-29	-107	9
-188	-112	-176	-114	-132	-40	-21	62	143	156	154	77	1920
6	-70	-42	-90	-104	-109	-53	-32	54	150	107	70	1
-34	-24	-80	-174	-173	-190	-140	-61	109	204	185	233	2
150	106	12	-76	-172	-187	-147	-158	-47	7	83	111	3
129	109	-10	-22	-84	-110	-102	-114	-53	-12	4	82	4
102	153	85	27	-6	-11	-31	10	64	43	83	81	1925
157	127	46	-6	-21	-64	-80	-88	-22	14	-43	44	6
25	38	59	31	-20	-39	-40	6	64	50	89	62	7
51	9	-27	-8	-114	-27	-56	6	17	34	7	-6	8
-17	-31	-72	-115	-138	-89	-17	61	99	104	137	100	9
50	-95	-124	-129	-182	-152	-113	-52	47	89	92	101	1930
18	-59	-109	-229	-209	-238	-198	-134	-58	102	114	126	1
84	4	-91	-180	-174	-201	-229	-188	-150	-22	105	131	2
125	67	3	-97	-141	-188	-193	-173	-96	-38	14	71	3
89	71	-4	-37	-55	-130	-177	-141	-100	-82	17	-5	4
10	25	17	-2	-15	-21	-21	-17	-9	9	19	9	1935
-29	-57	-74	-81	-76	-53	-18	20	53	74	81	33	6
-17	-61	-4	-11	-113	-100	-43	4	45	74	90	92	7
67	-14	-77	-119	-147	-145	-96	-43	18	81	130	127	8
95	40	-30	-64	-104	-126	-137	-54	-2	37	85	140	9
117	70	37	-35	-97	-140	-109	-56	-8	35	61	80	1940
25	8	10	-2	-42	-90	-112	-136	-124	-68	-71	-9	1
-9	-20	29	-31	-20	-35	-27	-29	-96	-109	-102	-137	2
-134	-76	-68	-45	-60	-7	42	32	54	30	-14	-51	3
-103	-144	-153	-141	-119	-90	22	90	168	180	134	78	4
19	53	-148	-219	-232	-236	-119	-36	65	183	217	187	1945
158	86	4	-118	-208	-267	-284	-257	-174	-59	25	87	6
172	132	92	-6	-90	-161	-190	-229	-216	-176	-158	-103	7
-30	54	63	123	96	62	40	-41	-90	-161	-247	-249	8
-259	-228	-157	-69	15	67	86	138	125	35	-114	-216	9
-267	-347	-339	-274	-203	-100	-2	117	186	229	200	29	1950
-118	-241	-366	-460	-411	-317	-201	-12	126	238	258	246	1
141	-65	-210	-348	-440	-468	-445	-352	-189	-5	119	229	2
225	171	69	-51	-210	-328	-400	-391	-407	-260	-77	41	3
189	224	240	141	87	-62	-173	-259	-315	-290	-271	-201	4

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Results . . . . . . Vol. VIII, pp. 170–172.
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Contribute Astronomico Capodimonte II, vol. IV no. 1, p. 7, no. 2, p. 6, no. 2, p. 18 and no. 6, p. 44. (These results are *not* definitive).
Typescript. Definitive results communicated to Dublin meeting of I.A.U. 1955.

TABLE 1b *Results of Variation of Latitude*

l

	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9
1890	−271	−258	−188	−73	55	154	216	202	90	−55
1	−200	−260	−260	−210	−70	65	184	257	254	186
2	61	−72	−189	−249	−228	−90	61	160	179	158
3	104	14	−68	−139	−197	−182	−98	9	94	144
4	126	84	34	−17	−53	−79	−108	−119	−91	−51
1895	−33	3	−7	−11	8	43	71	18	−43	−95
6	−115	−111	−84	−23	53	139	181	162	91	−8
7	−107	−186	−216	−177	−77	51	153	204	225	148
8	53	−65	−172	−202	−174	−105	27	158	199	151
9	91	26	−38	−70	−80	−104	−80	−17	63	12
1900	39	60	37	−8	−40	−39	−55	−62	−63	−53
1	−1	26	58	83	106	122	88	30	−25	−80
2	−102	−78	−51	38	134	205	200	139	44	−56
3	−144	−179	−142	−68	29	114	188	209	171	86
4	−43	−146	−170	−162	−94	1	87	151	181	148
1905	92	−11	−121	−144	−131	−82	1	84	122	148
6	97	24	−26	−71	−110	−109	−68	−33	−2	17
7	51	64	57	57	28	−4	−29	−84	−135	−115
8	−63	15	112	183	226	215	150	33	−104	−212
9	−270	−249	−171	−6	174	295	338	277	122	−56
1910	−185	−254	−295	−239	−73	121	275	323	300	199
1	36	−117	−208	−227	−167	−79	59	175	251	297
2	246	102	−22	−109	−140	−123	−91	−44	20	85
3	121	136	141	137	123	85	8	−60	−99	−100
4	−80	−20	81	171	197	192	151	70	−24	−112
1915	−174	−181	−109	0	115	223	284	271	164	28
6	−102	−167	−193	−152	−43	95	231	306	301	210
7	85	−30	−130	−151	−80	0	89	159	182	174
8	131	26	−33	−61	−76	−51	26	66	87	101
9	108	77	63	89	87	69	30	−27	−66	−63
1920	−21	28	80	138	198	213	197	147	72	−10
1	−76	−103	−55	42	153	238	250	210	140	61
2	−28	−100	−80	−8	92	191	272	217	167	75
3	−46	−108	−164	−131	−77	19	134	199	199	148
4	46	−36	−94	−102	−90	−55	44	94	107	90
1925	59	−4	−49	−65	−27	−22	24	60	71	53
6	8	−32	−90	−97	−87	−43	5	45	52	25
7	15	32	27	21	25	27	49	42	24	6
8	−44	−73	−65	−25	−13	−6	34	6	−49	−62
9	−67	−88	−61	14	64	104	114	86	−3	−65
1930	−113	−147	−102	−51	17	104	165	139	63	−6
1	−130	−150	−162	−95	−23	104	180	202	77	11
2	−89	−200	−197	−132	−94	−39	85	174	160	89
3	−3	−103	−134	−161	−117	−70	27	72	74	50
4	9	−49	−116	−167	−116	−54	14	36	40	33
1935	−14	4	12	8	11	52	20	19	−20	−46
6	−63	−66	−52	−14	49	98	98	67	12	−37
7	−94	−137	−96	−22	40	102	140	94	35	−25
8	−112	−133	−110	−70	−18	62	160	180	108	28
9	−85	−110	−116	−55	−10	54	140	179	166	98
1940	2	−68	−114	−134	−76	28	84	115	122	54
1	70	38	3	−13	−4	21	40	66	82	74
2	23	7	−4	1	9	9	−7	−8	2	14
3	30	56	87	118	145	129	104	63	7	−49
4	−66	−26	4	60	210	268	272	182	−10	−84
1945	−144	−173	−152	−60	92	191	280	305	252	115
6	12	−154	−193	−142	−74	−5	140	249	285	242
7	170	10	−94	−183	−194	−87	32	132	226	256
8	251	224	170	90	−28	−70	−75	−104	−49	114
9	68	178	244	248	220	160	54	−74	−152	−189
1950	−180	−105	34	174	282	361	347	170	−19	−204
1	−302	−332	−256	−58	140	302	445	408	260	58
2	−196	−358	−392	−298	−135	104	302	420	420	254
3	19	−129	−317	−372	−308	−140	48	245	336	283
4	100	0	−100	−180	−240	−140	−40	0	40	70

Sources:
1890·0–1899·8 Albrecht
1899·9–1905·9 Albrecht and Wanach
1906·0–1911·9 Wanach
1912·0–1922·6 Wanach and Mahnkopf
1922·7–1934·9 Kimura
1935·0–1940·0 Carnera
1940·0–1948·9 Carnera

1949·0–1954·9 Cecchini

					m					
·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	
83	−110	−246	−279	−221	−126	22	144	232	279	1890
240	100	− 50	−190	−240	−166	− 34	95	157	168	1
134	71	19	− 4	− 50	−156	−199	−160	− 32	46	2
107	121	111	87	43	− 11	− 71	−111	−112	− 78	3
− 49	− 16	12	43	95	123	118	70	10	− 51	4
− 87	−117	− 87	− 28	37	106	101	110	90	53	1895
− 29	−102	−139	−153	−140	− 95	11	110	222	195	6
117	22	− 83	−143	−181	−171	−139	− 73	− 7	130	7
147	107	55	− 10	− 87	−146	−176	−140	− 65	29	8
99	150	140	98	25	− 45	−119	−136	−118	− 73	9
− 30	15	40	57	78	76	50	6	− 29	− 42	1900
− 47	− 52	− 54	− 28	27	100	141	130	110	77	1
11	− 77	−149	−186	−180	−110	− 22	61	118	113	2
51	− 22	−110	−191	−240	−228	−165	− 51	76	143	3
141	86	− 5	− 97	−183	−211	−175	−123	− 55	28	4
101	144	125	54	− 38	−127	−182	−189	−145	− 68	1905
10	50	45	19	− 20	− 78	−113	−127	−130	−143	6
−119	− 60	− 16	15	53	75	67	22	− 41	−123	7
−192	−215	−198	−124	− 17	67	141	169	160	96	8
− 15	−152	−261	−312	−290	−174	− 8	127	228	282	9
237	94	− 73	−230	−328	−343	−266	−105	81	227	1910
288	239	122	− 36	−205	−321	−342	−286	−158	2	1
112	199	200	138	55	− 31	−109	−160	−171	−150	2
−100	− 48	10	57	91	109	111	91	24	− 46	3
− 99	−116	−114	−101	− 66	− 1	80	170	210	137	4
23	− 92	−205	−282	−260	−175	− 63	70	211	271	1915
235	129	1	−132	−260	−273	−144	− 10	112	212	6
283	271	154	28	− 99	−199	−207	−135	− 44	40	7
123	132	55	− 20	− 93	−132	−129	−100	− 70	− 61	8
− 60	− 57	− 38	− 3	22	30	30	20	− 9	− 62	9
−124	−159	−154	−119	− 55	20	90	149	165	134	1920
49	− 29	− 81	−110	− 99	− 59	3	80	130	107	1
39	− 40	−117	−185	−181	−128	− 60	98	187	210	2
199	128	34	− 71	−172	−172	−151	− 74	4	87	3
117	118	18	− 36	− 87	−107	−106	− 69	− 24	24	4
82	130	100	30	0	− 26	− 8	37	60	77	1925
109	142	64	5	− 33	− 70	− 87	− 38	14	− 43	6
28	38	44	32	− 22	− 40	− 14	42	65	78	7
64	29	− 6	− 42	− 72	− 45	− 18	16	33	5	8
− 10	− 24	− 62	−112	−130	− 65	24	81	113	131	9
83	− 28	−107	−143	−178	−139	− 79	19	83	97	1930
73	− 22	− 99	−207	−228	−216	−161	− 76	90	118	1
112	43	− 69	−168	−190	−212	−207	−159	− 29	100	2
129	96	18	− 94	−148	−190	−182	−118	− 41	23	3
75	76	17	− 26	− 77	−143	−157	−112	− 73	14	4
8	20	23	1	− 16	− 22	− 20	− 12	6	20	1935
3	− 44	− 70	− 80	− 76	− 44	1	44	72	79	6
20	− 40	− 86	−110	−113	− 89	− 18	34	71	90	7
88	29	− 60	−115	−149	−133	− 71	− 2	73	131	8
119	74	− 14	− 61	−107	−130	− 99	− 16	33	87	9
139	93	50	− 25	−100	−138	− 84	− 23	31	63	1940
− 16	13	12	− 6	− 49	− 93	−124	−121	− 81	− 47	1
− 18	− 6	− 1	− 16	− 27	− 28	− 34	− 76	−103	−120	2
− 46	−112	− 70	− 55	− 36	5	37	56	33	− 12	3
− 58	−125	−156	−144	−116	− 84	60	150	180	130	4
62	− 12	−135	−211	−237	−228	− 65	38	78	215	1945
176	129	18	−104	−215	−276	−276	−200	− 72	34	6
125	160	97	− 1	−100	−169	−214	−218	−182	−146	7
− 82	15	63	113	96	58	10	− 66	−160	−252	8
−283	−252	−181	− 74	21	83	128	131	35	−122	9
−237	−324	−348	−280	−186	− 72	74	176	230	190	1950
− 18	−182	−336	−460	−406	−286	−100	102	238	259	1
225	40	−170	−334	−447	−467	−403	−240	− 14	142	2
237	204	95	− 48	−228	−358	−413	−386	−254	− 60	3
95	190	200	140	40	−100	−210	−285	−290	−250	4

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Resultate . . . . Band V, pp. 180–182.
Ergebnisse . . . Band VI, p. 222.
Results . . . . . . Band VIII, p. 232.
Definitive results reported to Dublin meeting of the I.A.U. 1955.
Contribute Astronomico Capodimonte II, vol. IV no. 1, p. 10, no. 2, p. 11 and p. 20, and no. 6, p. 46. (These results are *not* definitive.)
Definitive results reported to Dublin meeting of I.A.U. 1955.

Nicolini, it appears that the observers are satisfied that the overlap in the programmes has been sufficient to minimize any inhomogeneity of the results because of these changes.

TABLE II

Stations used in International Programme

i	Station	Longitude (λ)	Dates of functioning
1.	Mizusawa	141° 8′ E.	1899 Dec. to date.
2.	Tschardjui	(i) 63° 29′ E.	1899 Dec. to 1909 July.*
	Tschardjui	(ii) 63° 35′ E.	1909 July to 1919 May.
	Kitab	(iii) 66° 53′ E.	1930 Nov. to date.†
3.	Carloforte	8° 19′ E.	1899 Dec. to 1943. 1946 to date.
4.	Gaithersburg	77° 12′ W.	1899 Dec. to 1914 Dec. 1932 Apr. to 1953 Aug.†
5.	Cincinnati	84° 25′ W.	1899 Dec. to 1915 Dec.
6.	Ukiah	123° 13′ W.	1899 Dec. to date.

* The site of the observatory was moved in 1909 July.

† Kimura did not include the results of these stations when observations recommenced there, but Carnera did so after 1935.

However, the star catalogue used from 1899 to 1935 was that of Cohn in which some of the declinations are in error. Since 1935, the more accurate Boss's catalogue has been used but, even in this, errors large enough to affect the accuracy of the results are known to exist.

The reliability of the recorded results might be affected by the fact that equal weight cannot really be given to each station. Because of their favourable climates, Carloforte and Ukiah usually contribute many more observations than the other stations. Between 1900 and 1924 from two to three thousand pairs of stars were observed at Carloforte every year, but in 1925, 1926 and 1927 the numbers of pairs observed were respectively only 1248, 928 and 1358. At that particular time only three stations, Mizusawa, Carloforte and Ukiah, were in operation, so the results for this period might well be less reliable than those for other periods. There have been other intervals when there have been relatively few observations from particular stations.

There is, finally, the "closing error" of the observations (see Melchior, **3**, p. 17). This has always been systematically negative and positive respectively at stations in the Northern and Southern Hemispheres but, following the modifications in the observing procedure made by Kimura in 1922·7, the error was reduced in magnitude.

We see, therefore, that the series of observations really cover a fairly large number of sub-intervals of time, each of which differs from the others in one or more of the number of stations in action, the star-programme used, the method of observing used and the method of reduction followed.

We might expect that the sub-intervals are sufficient in number for our statistical theory to remain approximately true despite this inhomogeneity, systematic irregularities arising in a sub-interval due to the organization of the programme persisting only for a short time and changing sufficiently frequently for us to neglect the resulting bias in the estimates of the harmonic components comprising the assumed systematic part of the variation (see **1**, equation (18)). In (**1**, p. 449), however, we suggested that the adequacy of our representation

of the systematic variation might be tested by dividing the series of observations into several consecutive sub-series and obtaining estimates from these which could be compared with estimates for the whole series. This no longer seems reasonable. If, on the one hand, we choose several short sub-series it seems probable that they would not include enough of the sub-intervals described above for our theory to be applicable. On the other hand, the inclusion of enough sub-intervals would give too few sub-series for useful comparisons to be made.

4. The computational procedure follows that described in our previous paper. There we discussed, at some length, the factors governing the choice of the lags which ought to be adopted in evaluating the covariances of the various series and argued (pp. 450–451) that it seems best in general to use the smallest possible lags. We have, however, made a number of analyses using higher lags; for these we need formulae which are slight generalizations of those given before. In (**1**, p. 450) equation (29) generalizes to

$$\cot r\gamma^* h = p(r)/q(r) \tag{3}$$

where $p(r)$ and $q(r)$ are as given in equation (30), r being the lag, $2\pi/\gamma$ the period and h the interval between observations. The damping factor, κ, is estimated from

$$e^{2\kappa^* h(r-1)} = \frac{\{p(1)\}^2 + \{q(1)\}^2}{\{p(r)\}^2 + \{q(r)\}^2}\,. \tag{4}$$

The equation

$$e^{2\kappa^* h(r_1-r_2)} = \frac{\{p(r_2)\}^2 + \{q(r_2)\}^2}{\{p(r_1)\}^2 + \{q(r_1)\}^2} \tag{4a}$$

may also be used, but detailed discussion of this is given later when we examine the question of bias (Sections 15 to 18).*

5. We have analysed the series and sub-series which are listed in Table III. They include unsmoothed and smoothed series in which the z-term has been taken into account and unsmoothed series in which the z-term has been ignored. The series have been given code numbers for ease of reference. The first digit of the code denotes the type of series (e.g. unsmoothed series in which the z-term has been taken into account are type-5 series), and the second digits distinguish the series according to the interval they cover.

As one of the major purposes of the work is to determine the period and damping factor of the free motion of the pole, values of these constants are given in the table, but it should be noted that the values given are those which have been obtained by using the smallest lags.

For the sake of completeness we have added the series which we considered in our previous paper, together with those considered by Jeffreys (**2**).

6. First we considered in more detail the data which we originally examined in (**1**). In the reduction of these observations, the z-term was ignored, and as it seems certain that this term is physically real we did not think it necessary to try to locate data for the years after 1934 (if indeed such data are to be found). The thirty-five year interval is covered by series 31. Series 32 cover the twenty-two complete years, 1900 Jan. to 1921 Dec. when the programme was directed at Potsdam and series 33 cover the twelve complete years 1923 Jan.

* An asterisked quantity is the estimated value of the unasterisked one, except that l^* and m^* are estimates of $l-l_s(=x)$ and $m-m_s(=y)$. In the tables asterisks are dropped because all the tabular values are estimates.

TABLE III

Description of data and principal results

Code	Length	Description	Free period ($2\pi/\gamma$) in years	Damping factor (κ) in years^{-1}
00	1890·0–1955·0	Smoothed; 0·1 year	1·304 ± 0·024	0·506 ± 0·095
01	1900·0–1934·9	interval; z-term taken	1·287 ± 0·028	0·361 ± 0·107
02	1900·0–1920·9	into account. Data of Table I*b*.	1·249 ± 0·025	0·198 ± 0·100
15	1892·0–1932·9	Smoothed; 0·1 year	1·223 ± 0·019	0·058 ± 0·015
16	1908·3–1921·5	interval; z-term taken into account. Data reduced by taking non-overlapping means of three consecutive values. Jeffreys's data.	1·202 ± 0·016	0·066 ± 0·007
31	1900 Jan. –1934 Dec.	Unsmoothed; monthly	1·267 ± 0·039	0·322 ± 0·103
32	1900 Jan. –1921 Dec.	interval; z-term *not*	1·248 ± 0·034	0·261 ± 0·113
33	1923 Jan. –1934 Dec.	taken into account.	1·355 ± 0·099	0·661 ± 0·340
34	1900 Jan. –1905 Dec.		1·241 ± 0·054	0·366 ± 0·221
50	1899 Dec.–1954 Dec.	Unsmoothed; monthly	1·287 ± 0·026	0·441 ± 0·095
51	1900 Jan. –1934 Dec.	interval; z-term taken	1·267 ± 0·041	0·356 ± 0·109
52	1900 Jan. –1920 Dec.	into account. Data of Table I*a*.	1·238 ± 0·033	0·239 ± 0·111

to 1934 Dec., during which Kimura was director. The change in the method of making the observations was made in 1922 July, so this year has been omitted. Series 33 should be homogeneous in every respect, but in 32, although the methods of reduction and observation remained unchanged, there were changes in the star-programmes and numbers of stations in operation. We have also examined series 34 which cover the first six years of the work of the International Service and should be homogeneous. The main results are given in Table IV. Series 33 appear to give results which are not in accord with the other three series of this group although it is unlikely that the difference would be highly significant. The physical explanation of this apparent anomaly is possibly to be found in the fact that the interval covered was short and seriously affected by the paucity of observations to which we have referred earlier. Series 34 give a value of the free period in reasonable agreement with the remaining two but the value of κ^* obtained from these series appears to be subject to considerable uncertainty. In this case, the fact that λ^* exceeds unity by more than three times its estimated standard error may well be attributed to the inapplicability of the asymptotic theory (developed in Sections 4 and 5 of I) to such short series. (It will be recalled that λ is the parameter that measures the effect of observational error.)

7. Next we examined the smoothed series in which account was taken of the z-term. We have analysed the longest available series, 00, covering the sixty-five years from 1890·0 to 1954·9 and also the series 01 and 02 which correspond to the series 31 and 32. The results for the type-0 series are given in Table V.

In this group of results there is a striking decrease of both the estimated values of the free period and the damping factor as the series shorten. The results are not particularly consistent with each other, especially in the case of values of κ^*.

TABLE IV

Results for series of type-3

(*a*) Harmonic components of systematic annual motion (unit 0″·001)

	Series 31			Series 32		Series 33		Series 34		
	l	m	est. stand. error	l	m	l	m	l	m	est. stand. error
c	+11·06	−10·06	± 3·05	+18·57	− 3·59	− 7·59	−21·23	+ 0·32	+11·90	± 6·01
a_1	−55·11	+74·79	± 10·77	−52·35	+63·91	−53·78	+92·28	−42·35	+61·96	± 24·35
b_1	−70·29	−45·28		−61·57	−39·48	−82·42	−46·03	−67·64	−28·72	
a_2	− 2·14	− 5·55	± 2·40	− 2·34	− 8·51	− 1·11	+ 2·31	− 1·96	− 2·44	± 4·51
b_2	− 6·91	− 3·11		+ 7·80	− 5·11	+ 4·11	+ 2·53	+ 6·28	− 5·29	
a_3	+ 2·35	+ 2·24	± 1·70	+ 3·21	+ 2·20	+ 1·06	+ 4·40	− 0·58	− 1·22	± 2·76
b_3	− 4·01	+ 3·94		− 1·37	− 4·42	− 8·71	+ 3·50	− 1·83	− 1·14	
a_4	− 1·76	+ 1·02	± 1·48	− 1·95	+ 1·99	− 1·31	+ 0·81	− 1·07	+ 2·06	± 2·15
b_4	+ 1·11	+ 1·22		+ 0·20	+ 2·95	+ 2·61	− 1·00	− 0·79	+ 0·48	
a_5	− 0·91	− 1·00	± 1·39	− 2·23	+ 2·47	+ 0·97	+ 0·72	− 4·73	− 0·40	± 1·90
b_5	+ 2·09	+ 1·26		+ 1·95	+ 1·89	+ 1·83	+ 2·44	− 0·75	+ 3·33	
a_6	− 0·77	− 1·56	± 0·97	− 0·28	− 1·89	− 1·74	+ 0·27	− 0·29	− 3·68	± 1·83

(*b*) Free Motion

	Series 31	Series 32	Series 33	Series 34
γh	23° 40′ ± 44′	24° 2′ ± 39′	22° 8′ ± 1° 37′	24° 10′ ± 1° 3′
period	1·267 ± 0·039 years	1·248 ± 0·034 years	1·355 ± 0·099 years	1·241 ± 0·054 years
κ	0·322 ± 0·103 year^{-1}	0·261 ± 0·113 year^{-1}	0·661 ± 0·340 year^{-1}	0·366 ± 0·221 year^{-1}
λ	0·9803 ± 0·0035	0·9939 ± 0·0025	0·9639 ± 0·0149	1·013 ± 0·004
σ	24·6	25·83	20·56	21·14
σ'	15·3	9·72	12·32	...

The most serious feature of these results is that $\lambda^*>1$. Taking $\lambda=\lambda^*$ has little effect on the estimate of $\mathrm{var}\,\kappa^*h$, but makes a great deal of difference to those of $\mathrm{var}\,\lambda^*$ and $\mathrm{var}\,\gamma^*h$, the latter being in fact useless since it is then negative. The calculation of estimated variances is particularly suspect for this group of smoothed series in view of the values of the ratio of λ^*-1 to $\sqrt{\{(\mathrm{var}\,\lambda^*)_{\lambda=1}\}}$. These are 12·6 for series 00, 10·4 for series 01, and 10·3 for series 02, all of which are so highly significant as to make the results unacceptable. The discrepancy must of course be due to the behaviour of the auto-covariances and cross-covariances with lags 0, 1, 2 since only these are used in calculating the ratio.

Since the three series overlap, the estimates of γh, κ and σ are related to each other; because of this it is very troublesome to carry out even approximate large sample tests for the significance of the differences between the estimates. The extent of the correlation could be worked out but the calculations would be so tedious that we do not think they would be justified, in view of the behaviour of λ^*.

Table V

Results for series of type-0

(*a*) Harmonic components of systematic annual motion (unit 0″·001)

	Series 00 l	Series 00 m	Series 01 l	Series 01 m	Series 02 l	Series 02 m
c	+17·81	−28·10	+19·00	−20·92	+27·40	−22·33
a_1	−36·51	+75·25	−39·76	+79·87	−41·12	+66·92
b_1	−89·30	−28·97	−77·36	−30·69	−64·80	−31·24
a_2	− 0·90	− 0·36	+ 1·05	− 1·33	+ 3·18	− 2·43
b_2	+ 1·32	− 0·70	+ 1·68	+ 0·25	+ 2·19	+ 0·06
a_3	+ 0·61	+ 0·36	+ 2·62	− 0·96	+ 1·80	− 1·03
b_3	− 1·93	+ 0·78	− 1·50	+ 1·42	− 0·08	− 0·11
a_4	− 1·58	− 0·49	− 1·15	− 0·04	0·00	− 1·35
b_4	+ 0·06	+ 0·55	− 0·21	+ 0·53	+ 0·22	+ 0·61
a_5	− 0·69	− 0·03	+ 0·01	− 0·23	+ 0·45	− 0·59

(*b*) Free Motion (calculated with least possible values of lag)

	Series 00	Series 01	Series 02
γh	27° 37′ ± 31′	27° 58′ ± 36′	28° 50′ ± 34′
period	1·304 ± 0·24 years	1·287 ± 0·028 years	1·249 ± 0·025 years
κ	0·506 ± 0·095 year^{-1}	0·361 ± 0·107 year^{-1}	0·198 ± 0·100 year^{-1}
λ	1·037 ± 0·003	1·029 ± 0·003	1·020 ± 0·002
σ	37·3	28·6	25·0
σ'	...	...	...

N.B. The standard errors were calculated using $\lambda=1$.

8. Finally, we examined the unsmoothed series in which account was taken of the z-term. The longest suitable series, 50, cover the fifty-five years 1900 Jan. to 1954 Dec. Series 51 and 52 correspond to series 31 and 32 respectively. The results derived from this group of type-5 series are given in Table VI. Here we see again that the estimates of the length of the free period and of the damping factor decrease with the length of the interval. This group of results shows much more consistency than do either the groups of type-0 or type-3 series and there is no trouble over the values of λ^* which are all significantly less than unity.

Again the overlapping of the intervals makes comparisons difficult but the estimates of the standard errors are now acceptable and so do give a rough idea of the consistency of the estimates. We can use the inequality $\sigma(X-Y)\leqslant\sigma(X)+\sigma(Y)$

(for any random variables X, Y) to obtain an under-estimate of the significance of a difference—though admittedly this is very much of an under-estimate if there is a fair degree of positive correlation between the two variables. Thus we would probably be justified in concluding that the difference between the λ^*'s for series 50 and 51 was significant since this is 0·0182 compared with an estimated standard error of *at most* 0·0022 + 0·0038 = 0·0060.

Table VI

Results for series of type-5

(*a*) Harmonic components of systematic annual motion (unit 0″·001)

	Series 50			Series 51			Series 52		
	l	m	est. stand. error	l	m	est. stand. error	l	m	est. stand. error
c	+18·80	−31·72	± 3·17	+12·92	−22·02	± 3·20	+19·00	−23·30	± 3·86
a_1	−64·45	+69·51	± 10·58	−55·33	+74·99	± 11·36	−48·42	+65·67	± 15·75
b_1	−71·41	−45·59		−69·59	−45·89		−59·68	−37·05	
a_2	− 1·29	− 4·91	± 2·28	− 2·15	− 6·00	± 2·51	− 2·51	− 7·95	± 3·00
b_2	+ 5·66	− 0·39		+ 6·79	− 3·05		+ 8·37	− 5·45	
a_3	− 0·16	+ 1·38	± 1·44	+ 1·84	+ 2·36	± 1·77	+ 2·91	+ 2·29	± 1·96
b_3	− 2·59	+ 2·79		− 4·15	+ 3·93		− 1·19	+ 4·38	
a_4	− 0·78	+ 1·47	± 1·14	− 1·02	+ 0·55	± 1·53	− 0·65	+ 0·89	± 1·62
b_4	+ 1·62	+ 1·52		+ 1·25	+ 1·36		+ 0·81	+ 4·28	
a_5	+ 0·21	+ 0·77	± 1·02	+ 0·58	+ 0·41	± 1·45	− 1·85	+ 1·85	± 1·49
b_5	+ 1·59	+ 0·63		+ 2·12	+ 1·77		+ 2·11	+ 2·94	
a_6	− 0·71	− 0·85	± 0·70	− 0·72	− 1·63	± 1·00	+ 0·19	− 2·65	± 1·03

(*b*) Free Motion (calculated with least possible values of lag)

	Series 50	Series 51	Series 52
γh	23° 18′ ± 28′	23° 41′ ± 46′	24° 14′ ± 39′
period	1·287 ± 0·026 years	1·267 ± 0·041 years	1·238 ± 0·033 years
κ	0·441 ± 0·095 year^{-1}	0·356 ± 0·109 year^{-1}	0·239 ± 0·111 year^{-1}
λ	0·9975 ± 0·0022	0·9793 ± 0·0038	0·9942 ± 0·0024
σ	32·4	25·8	25·2
σ'	6·0	15·6	9·8

9. The components of the systematic annual motion shown in Tables IV, V and VI vary according to type of series and interval, there being noticeable differences in both the amplitudes and phases of the several components. When the tests of significance used in (**1**) were applied, we found that on the whole only the annual and semi-annual components are highly significant. When we have actually evaluated the estimated standard errors of the components, they appear to be relatively large. The differences of corresponding harmonic components of type-5 series are given in Table VII, together with the estimated standard errors of the pairs of components. The overlapping of the three series has again the effect of making any precise assessment of the significance of the difference very troublesome, but in view of the magnitudes of the estimated standard errors no startling discrepancy is indicated, although the differences 50–51 and 50–52 for the constant terms in the m components are suggestive.

10. There may be systematic trends in the observational data caused by genuine secular trends in the systematic terms in F_1 and G_1 (cf. **1**, equation (3)).

TABLE VII

Comparison of harmonic components of type-5 series

Differences (l-components)
(unit $0''\cdot001$)

	50–51				50–52				51–52			
r	a_r	Est. s.e.'s		b_r	a_r	Est. s'e.'s		b_r	a_r	Est. s.e.'s		b_r
0	5·88	3·17,	3·20		− 0·20	3·17,	3·86		−6·08	3·20,	3·86	
1	−9·12	10·58,	11·36	−1·82	−16·03	10·58,	15·75	−11·73	−6·91	11·36,	15·75	−9·91
2	0·86	2·28,	2·51	−1·13	1·22	2·28,	3·00	− 2·71	0·36	2·51,	3·00	−1·58
3	−2·00	1·44,	1·77	1·56	− 3·07	1·44,	1·96	− 1·40	−1·07	1·77,	1·96	−2·96
4	0·24	1·14,	1·53	0·37	− 0·13	1·14,	1·62	0·81	−0·37	1·53,	1·62	0·44
5	0·79	1·02,	1·45	−0·53	2·06	1·02,	1·49	− 0·52	1·27	1·45,	1·49	0·01
6	0·01	0·70,	1·00		− 0·90	0·70,	1·03		−0·91	1·00,	1·03	

Differences (m-components)
(unit $0''\cdot001$)

	50–51				50–52				51–52			
r	a_r	Est. s.e.'s		b_r	a_r	Est. s.e.'s		b_r	a_r	Est. s.e.'s		b_r
0	−9·70	3·17,	3·20		− 8·42	3·17,	3·86		1·28	3·20,	3·86	
1	−5·48	10·58,	11·36	0·30	3·84	10·58,	15·75	− 8·54	9·32	11·36,	15·75	−8·84
2	1·09	2·28,	2·51	2·66	3·04	2·28,	3·00	5·06	1·95	2·51,	3·00	2·40
3	−0·98	1·44,	1·77	−1·14	− 0·91	1·44,	1·96	− 1·59	0·07	1·77,	1·96	−0·45
4	0·92	1·14,	1·53	0·16	0·58	1·14,	1·62	− 2·76	−0·34	1·53,	1·62	−2·92
5	0·36	1·02,	1·45	−1·14	− 1·08	1·02,	1·49	− 2·31	−1·44	1·45,	1·49	−1·17
6	0·78	0·70,	1·00		1·80	0·70,	1·03		1·02	1·00,	1·03	

We have examined the series of type-5 for such trends by fitting curves of the type

$$l_s = c_0 + c_1 t + a \cos \nu t + b \sin \nu t \qquad (2\pi/\nu = 1 \text{ year})$$

to each interval covered by only one director or only one star-programme. The results give values of c_0 markedly different from those of c recorded in Table VI and the values of c_1 are so large that the removal of systematic trends of the kind assumed here yields series for the free motion which are visibly irregular, with large discontinuities at each change in star-programme. If the trends are real, we cannot explain them satisfactorily. However, the frequent changes in programme make it difficult to determine trends accurately, and the large discontinuities found at each change in star-programme are likely to be due more to accidental rather than true physical causes.

11. We have examined the residual motion $\{\epsilon'^*, \eta'^*\}$ which includes the effects of both observational errors and the random disturbances which maintain the free motion. We have restricted our attention to those of type-5 series. The residuals have been estimated in the way described in (1, p. 458) for each of the series 50, 51 and 52, those for 50 being displayed in Fig. 1 (*b*, *d*).

If the series were heterogeneous and corrections such as those given by (1) should be applied, then the numerical harmonic analysis would leave in l^*, m^* errors which remain constant through each homogeneous sub-interval, but which change between such sub-intervals. As a result $\{\epsilon'^*\}$ and $\{\eta'^*\}$ would also be affected by similar errors. We have therefore determined the means of $\{\epsilon'^*\}$ and $\{\eta'^*\}$ over various sub-intervals during which star-programme and

Table VIII

Mean residuals : series of type-5 (unit 0″·001)

	Interval	$\bar{\epsilon}'$	$\bar{\eta}'$
Series 50			
Star Programme			
I	1900–05	2·89 ± 2·94	6·89 ± 2·90
II	1906–11	− 6·00 ± 3·22	5·39 ± 3·03
III	1911–22 July	8·83 ± 2·60	− 7·00 ± 3·38
I–III	1900–22 July	3·31 ± 1·71	− 0·02 ± 1·97
IV	1922 July–34	2·67 ± 2·60	9·89 ± 2·67
V	1935–48	3·73 ± 2·30	− 5·24 ± 2·24
VI	1949–54	−24·15 ± 5·44	− 9·86 ± 4·42
I–VI	1900–54	0·28 ± 1·28	− 0·19 ± 1·28
Series 51			
Star Programme		$\bar{\epsilon}'$	$\bar{\eta}'$
I		− 0·06 ± 2·89	3·58 ± 2·93
II		− 9·11 ± 3·12	2·21 ± 2·94
III		5·90 ± 2·59	−10·43 ± 3·35
I–III		0·33 ± 1·70	− 3·35 ± 1·95
IV		− 1·57 ± 2·71	6·17 ± 2·65
I–IV		− 0·35 ± 1·46	0·03 ± 1·59
Series 52			
Star Programme		$\bar{\epsilon}'$	$\bar{\eta}'$
I		− 0·26 ± 2·86	6·35 ± 2·93
II		− 9·57 ± 3·07	5·08 ± 2·88
III		5·44 ± 2·80	− 6·60 ± 3·41
I–III		− 0·48 ± 1·73	0·44 ± 1·91

direction remained uniform. The results are given in Table VIII. The uncertainties given in the table are the standard errors calculated in the normal way but, because of correlations introduced by the processes by which the estimated annual and forced motions were removed, and possibly also correlations between the true residuals ϵ'_n and ϵ'_{n+1} and η'_n and η'_{n+1}, they are not unbiased estimates of the true standard errors. The bias is very troublesome to calculate but may be expected in all cases to be positive, so that they are useful as estimates of upper limits to the uncertainties. The results suggest that the data do suffer from inhomogeneity.

12. When the results are grouped according to the intervals covered by the series it is immediately evident that the results for the group 02, 32 and 52 are very consistent, and those for the group 01, 31 and 51 are slightly less so. The agreement for the longer series 00 and 50 appears to be less good.

The evidence of the statistical analysis so far presented does not give any conclusive reasons for preferring or rejecting any particular set of results, although these for type-0 series are suspected to be unacceptable because $\lambda^* > 1$. When the results are considered in conjunction with the known facts about the initial data it seems reasonable to draw certain inferences.

The data might be inhomogeneous, and if the corrections given in (1) are applicable in full the long series might be expected to yield unreliable results. The results for series 00 and 50 seem to support this, and indeed, the results derived from series 50 appear to be less untypical compared with the main body of results than do those for series 00 which differ from 50 by the inclusion of the results of the uncoordinated observations made in the decade preceding the beginning of the international programme.

It might also be expected that the series covering the first twenty-one years of the international programme would give better results than those covering the first thirty-five years. We have already recorded that the differences in the constant terms of the harmonic analyses of the m components between the series 50 and 51 and between the series 50 and 52 are suggestive, and although these differences are not what they should be if they were due to the non-application of corrections of the order given by (1) they still support the opinion that the data which we have analysed are inhomogeneous to some extent.

It also appears that the variability of the results depends more on the interval covered by the series than on the type of series. The fact that $\lambda^* > 1$ in the case of type-0 series does suggest that smoothing has had an effect, but the differences between the results of type-3 and type-5 series suggest that the inclusion or exclusion of the z-term has had little influence. On observational grounds, however, the z-term must be accounted for and we conclude that the most reliable results are those derived from type-5 series.

The results of the analysis of the short series 33 and 34 give periods and damping factors which seem unsatisfactory, but their estimated standard errors are large enough to account for the discrepancies. The low precision of the estimates and the probable inaccuracy of the asymptotic theory for short series are likely to be general and we have not made any other analyses of such short series. This has led to the disadvantage, already noted, that the intervals are overlapping, but that has not proved to be too serious a defect.

13. Jeffreys's results are given in Table III as series 15 and 16. He used essentially the same data as those of our type-0 series. The periods which he

obtained are in fair agreement with those for our series 01 and 02 but the damping factors are markedly different. His value of λ for series 15 was less than unity. However his method of computing κ and λ was different from ours, being equivalent to solving (in our notation)

$$\lambda^2 e^{-2\kappa h r} = \frac{\{p(r)\}^2 + \{q(r)\}^2}{\{p(0)\}^2}$$

by least squares using three values of the lag r. He advocated the use of fairly large values of r, whereas we have argued that the smallest possible lags should give greatest accuracy (**1**, p. 450). Jeffreys also reduced the volume of data by taking non-overlapping means of three consecutive observations so that his lags 0, 1 and 2 correspond roughly to our 0, 3 and 6.

In order to make a direct comparison with Jeffreys's work we have examined two series, 05 and 06, for the intervals 1892·0 to 1932·9 and 1908·3 to 1921·5 which were the intervals of Jeffreys's series 15 and 16 in Table III. The data differ only in that we have not grouped them. We have calculated covariances only for every third lag and have used equation (4) in the modified form

$$e^{6\kappa^* h(r-1)} = \frac{\{p(3)\}^2 + \{q(3)\}^2}{\{p(3r)\}^2 + \{q(3r)\}^2} \tag{5}$$

and (4 *a*) with $r_1 = 3r$ and $r_2 = 3r-3$, so that

$$e^{6\kappa^* h} = \frac{\{p(3r-3)\}^2 + \{q(3r-3)\}^2}{\{p(3r)\}^2 + \{q(3r)\}^2}. \tag{5 a}$$

The results given in Table IX show that for small lags, the estimates of the period agree favourably with those given by Jeffreys so it seems that his grouping of the data has had no serious effect. The damping factors, except those for series 06 computed by formula (5), seem to be irregular, but for both period and damping factor it can be said that as the lag increases there is initially a marked decrease in the estimates followed by a more or less marked oscillation.

Table IX

Analysis of series 05 and 06 for various lags

$3r$	Period in years		Damping factor in year^{-1}			
	Series 05	Series 06	Series 05		Series 06	
			Formula (5)	Formula (5*a*)	Formula (5)	Formula (5*a*)
3	1·230	1·216				
6	1·193	1·206	+0·169	+0·169	+0·064	+0·064
9	1·178	1·202	+0·037	−0·096	+0·028	−0·008
12	1·190	1·201	−0·005	−0·088	+0·024	+0·017
15	1·209	1·201	+0·036	+0·159	+0·022	+0·015
18	1·189	1·203	+0·060	+0·159	+0·022	+0·021
21	1·184	1·202	+0·035	−0·090	+0·022	+0·022
24	1·187	1·200	+0·021	−0·063	+0·021	+0·012

14. As we believe that the data of the type-5 series are superior for the purposes of analysis, we have carried out more calculations for the series 50, 51 and 52. The results are given in Table X. (Formula (4 *a*) was used with $r_1 = r$, $r_2 = r-1$.)

The estimated period is clearly seen to be a damped oscillatory function of the lag. For the damping, the estimates follow a similar pattern but there is an appreciable difference between the results according to which formula is used.

Table X

Analysis of series 50, 51 *and* 52 *for various lags.*

	Period in years			Damping factor in year^{-1}					
	Series	Series	Series	Series 50		Series 51		Series 52	
r	50	51	52	Form. 4	Form. 4*a*	Form. 4	Form. 4*a*	Form. 4	Form. 4*a*
1	1·288	1·268	1·237						
2	1·265	1·253	1·227	0·443	+0·443	0·352	+0·352	0·231	+0·231
3	1·241	1·239	1·219	0·457	+0·471	0·379	+0·406	0·241	+0·251
4	1·220	1·226	1·212	0·436	+0·394	0·389	+0·409	0·234	+0·220
5	1·205	1·215	1·207	0·400	+0·291	0·387	+0·380	0·223	+0·190
6	1·194	1·203	1·203	0·348	+0·142	0·350	+0·200	0·199	+0·105
7	1·185	1·193	1·199	0·315	+0·150	0·315	+0·140	0·174	+0·045
8	1·178	1·186	1·196	0·270	−0·003	0·254	−0·107	0·138	−0·074
9	1·172	1·181	1·194	0·221	−0·118	0·200	−0·184	0·105	−0·127
10	1·169	1·179	1·193	0·166	−0·276	0·151	−0·239	0·077	−0·152
11	1·169	1·178	1·192	0·117	−0·318	0·109	−0·263	0·051	−0·180
12	1·171	1·180	1·193	0·083	−0·257	0·077	−0·243	0·034	−0·138
13	1·177	1·183	1·194	0·066	−0·128	0·066	−0·057	0·033	+0·024
14	1·182	1·185	1·195	0·070	+0·118	0·069	+0·097	0·039	+0·114
15	1·185	1·187	1·195	0·080	+0·216	0·075	+0·165	0·045	+0·126
16	1·187	1·189	1·195	0·092	+0·254	0·081	+0·158	0·050	+0·108

15. Professor Jeffreys has suggested to us that the use of higher lags may be advisable because of the possibility that the disturbances ϵ and η are not uncorrelated as we assume in our model. There are good physical reasons for believing that correlations between the disturbances do exist; e.g. the seasonal climatic changes which give rise, at least in part, to the systematic terms $F_s(t)$ and $G_s(t)$ (1, equation (4)) are not entirely smooth, and irregularities in them tend to persist over intervals of several weeks, so that there may well be appreciable positive correlation between values of ϵ or η at times separated by intervals up to 2 or 3 months.

Such correlation may cause our estimates of the period and of the damping factor to be biased. We have investigated the bias on the assumption that $\mathrm{cov}(\epsilon_t, \epsilon_{t+rh}) = \mathrm{cov}(\eta_t, \eta_{t+rh}) = g_r$, say, while $\mathrm{cov}(\epsilon_t, \eta_{t+rh})$ remains equal to zero for all r. It is then easily shown that

$$b(r\gamma^* h) \doteqdot \tan^{-1}\frac{B(r)}{A(r)} - r\gamma h \tag{6}$$

and for the estimate κ^* given by equation (4*a*)

$$b\{(r_1 - r_2)\kappa^* h\} \doteqdot \log\left|\frac{e^{r_2\kappa h}\{A(r_2) + iB(r_2)\}}{e^{r_1\kappa h}\{A(r_1) + iB(r_1)\}}\right| \tag{7}$$

where

$$A(r) + iB(r) = \sum_{s,t=0}^{\infty} e^{-\kappa h(s+t) + i\gamma h(s-t)} g_{r-s+t}$$

and we use the notation $b(\theta^*)$ for the bias in an estimate θ^* of a quantity θ.

In particular, when $g_r = g_0\mu^r$ $(r > 0)$, where $|\mu| < 1$, i.e. the auto-correlation function of the series $\{\epsilon_t\}$, $\{\eta_t\}$ is that of a first-order autoregressive process,

(6) and (7) give

$$b(r\gamma^* h) \doteqdot \frac{-2\mu\kappa h \sin\gamma h}{1+\mu^2-2\mu\cos\gamma h} + \left(\frac{2\kappa h\mu^{r+1}}{1-\mu^2}\right) \sin\{(r+1)\gamma h - \phi\} \tag{8}$$

and

$$b\{(r_1-r_2)\kappa^* h\} \doteqdot \frac{2\kappa h}{1-\mu^2}\left[\mu^{r_1+1}\cos\{(r_1+1)\gamma h-\phi\} - \mu^{r_2+1}\cos\{(r_2+1)\gamma h-\phi\}\right] \tag{9}$$

where

$$\phi = 2\tan^{-1}\left(-\frac{\mu\sin\gamma h}{1-\mu\cos\gamma h}\right)$$

and it is assumed that r, r_1, r_2 are sufficiently small for $e^{r\kappa h}$, $e^{r_1\kappa h}$, $e^{r_2 h}$ to be taken approximately equal to unity. Thus as r increases the bias in γ^* tends to zero like $1/r$ after an initial damped oscillation. When κ^* is calculated by formula (4) the bias is given by (9) with $r_1 = r$ and $r_2 = 1$. This behaves in the same way as the bias in γ^*. When κ^* is calculated by formula (4 *a*), however, the bias is given by (9) with $r_1 = r$ and $r_2 = r-1$ and oscillates with an amplitude that decreases like μ^r. The behaviour of the biases should in general not be very different from this since $g_0, g_1, g_2, \ldots$ may be expected to be a decreasing sequence of positive numbers. In fact, if g_r can be put equal to zero for $r > q$, it follows from (6) and (7) that for $r > q$, the biases of γ^* and κ^* calculated from equation (4) are proportional to $1/r$, while the bias in κ^* calculated from equation (4 *a*) with $r_1 = r$, $r_2 = r-1$ is negligibly small. This should be the case for $q = 3$ or 4 since correlations between the disturbances should not extend over intervals exceeding a few months.

We conclude that correlation of this type would suffice to explain the behaviour of the estimates of $T = 2\pi/\gamma$ and κ in Table X for small values of r. However, it cannot reasonably be considered responsible for the persistence of oscillations for the larger values of r (those of κ^* in the last three columns being particularly marked), since that would require the auto-correlations g_r/g_0 to remain appreciable for these values of r.

16. The imperfect removal of systematic variations from the residual series used for the calculation of $p(r)$ and $q(r)$ will also produce bias in our estimates. Suppose that $\{\xi_t, \zeta_t\}$ represent systematic variations which have not been allowed for in determining the free motion. Then $p(r)$ and $q(r)$ are calculated from $\{X_t, Y_t\}$ instead of $\{x_t, y_t\}$ where

$$X_t = x_t + \xi_t, \qquad Y_t = y_t + \zeta_t.$$

If this is so, it can easily be shown that the consequential biases in $p(r)$ and $q(r)$ are

$$b(p(r)) = \frac{1}{N-r}\sum_t (\xi_t\zeta_{t+rh} + \zeta_t\zeta_{t+rh}), \tag{10}$$

$$b(q(r)) = \frac{1}{N-r}\sum_t (\xi_t\zeta_{t+rh} - \xi_{t+rh}\zeta_t), \tag{11}$$

N being the total number of observations.

The results of Section 9 suggest that the annual term is different in different sub-intervals so that the process which we adopted of removing a uniform annual term throughout the whole interval must lead to the presence of terms ξ_t and ζ which take the form

$$\xi_t = a_0 + a_1\cos(\nu t + \phi),$$
$$\zeta_t = b_0 + b_1\cos(\nu t + \psi),$$

where a_0, a_1, b_0, b_1, ϕ and ψ are constants and $2\pi/\nu = 1$ year; there may also be small harmonic terms. In the true situation, the constants change from one sub-interval to another, but for simplicity we will first consider them unchanged over the whole interval. We find that the biases in $p(r)$ and $q(r)$ lead to

$$b(r\gamma^*h) \doteqdot \frac{-e^{r\kappa h}\sin r\gamma h}{2V_{11}(0)}(a_0^2+b_0^2+\tfrac{1}{2}(a_1^2+b_1^2)\cos\nu rh) \tag{12}$$

provided that λ is nearly equal to unity. ($V_{11}(0)$ is the expected value of $\frac{1}{2}p(0)$ in the absence of bias—see (**1**, equation (14)). If the damping is estimated by using equation (4), the bias is given by

$$\begin{aligned} b\{2(r-1)\kappa^*h\} \doteqdot \{&\cos\gamma h[a_0^2+b_0^2+\tfrac{1}{2}(a_1^2+b_1^2)\cos\nu h] \\ &-e^{(r-1)\kappa h}\cos r\gamma h[a_0^2+b_0^2+\tfrac{1}{2}(a_1^2+b_1^2)\cos\nu rh]\}/V_{11}(0) \end{aligned} \tag{13}$$

and if it is obtained by using equation ($4a$) with $r_1 = r_2 + 1 = r$, it is given by

$$\begin{aligned} b\{2\kappa^*h\} \doteqdot \frac{e^{(r-1)\kappa h}}{V_{11}(0)}\{&(a_0^2+b_0^2)[\cos(r-1)\gamma h-\cos r\gamma h] \\ &+\tfrac{1}{2}(a_1^2+b_1^2)[\cos(r-1)\gamma h\cos\nu(r-1)h-\cos r\gamma h\cos\nu rh]\}. \end{aligned} \tag{13a}$$

In each of these formulae we have omitted "cross-product" terms involving $a_1b_1\sin(\phi-\psi)$ which appear in the detailed working. Their inclusion would, at worst, double the estimated bias, but should have a much smaller effect.

If the period $2\pi/\gamma$ is denoted by T, then

$$\frac{b(T^*)}{T} \doteqdot -\frac{b(r\gamma^*h)}{r\gamma h},$$

and the bias in T, regarded as a function of r, is an oscillating one with an amplitude which varies with $e^{r\kappa h}/r$, which decreases as r increases, at least initially.

For the damping, formula (4) leads to a bias in κ^* which consists partly of a term decreasing with $1/(r-1)$ and partly of an oscillation with an amplitude which decreases (initially) with $e^{(r-1)\kappa h}/(r-1)$. Formula ($4a$) leads to an oscillatory bias whose amplitude varies with $e^{(r-1)\kappa h}$.

The formulae given are not valid for very large lags, when the second order terms, which have been neglected, may become more important than the terms given.

When this analysis is extended to deal with ξ_t and ζ_t whose amplitudes change from sub-interval to sub-interval, the results are similar but, because of the increasing overlap of sub-intervals as r increases, the biases tend to decrease more and for sufficiently long series with sufficiently many sub-intervals the biases could become negligible—thus giving a justification for a remark made to this effect in Section 3 above.

17. Values of a_0, b_0, a_1 and b_1 consistent with the corrections given by (1) yield values of $b(p(r))$ which are comparable with $p(r)$ itself, and our results do not indicate such serious effects. However, the results of harmonic analysis given in Tables IV, V and VI show that the inhomogeneity could give values of the constants of order of magnitude 10 units (the unit being $0''\cdot001$). Putting $a_0 = b_0 = a_1 = b_1 = 10$ units, the largest $b(p(r))$ is about 50 units and hence the largest bias in $r\gamma^*h$ is about 0·02 in the range of values of r up to $r = 16$. The results correspond to a bias in the period which is approximately $0{\cdot}06/r$ years, which is in rough agreement with the behaviour of our estimates of T. Again,

using formula (4), the bias in κ^* given by the above analysis is at most about $0{\cdot}36/(r-1)$ which also compares with the behaviour of the estimates of κ.

18. It can also be shown that other kinds of systematic inhomogeneity in the data, not necessarily directly connected with the amplitudes of the annual terms, lead to biases similar to that arising from the sinusoidal terms (ξ_t, ζ_t). For example, systematic errors in individual star-places could give rise to a systematic contribution to the observational error which would repeat annually until the particular stars were replaced at a subsequent change in the observing programme; such an error would contribute to the bias in much the same way as considered above.

We may conclude that the behaviour of our estimates is explained by biases introduced by inhomogeneity and auto-correlation of the disturbing function.

19. We believe that the sub-intervals during which the data can be assumed to be homogeneous are too short ever to give a reasonable prospect of eliminating the systematic motion adequately; this is really a problem which is better tackled in the design of the observing programme rather than in the subsequent analysis of the results. Despite the complexities introduced by the inhomogeneity and the auto-correlation of the disturbances, however, it seems practicable to make a reasonable estimate of the period. First we have to detect a lag beyond which it is unlikely that auto-correlation of the disturbances is effective. Secondly, considering the form of $b(r\gamma^*h)$ in equation (12), we must judge where the oscillations have decreased to an extent such that a mean value is well determined. Inspection of the results in Table X suggests that both these requirements are met by taking $r \geqslant 6$. The maximum variation in the estimated period thereafter is less than about 10 days for series 50 and 51 and 4 days for series 52. The means of the estimated values of $T = 2\pi/\gamma$ for $r = 1$ to 16 and $r = 6$ to 16 are given below.

	Series 50	Series 51	Series 52
$r=1$ to 16	1·198 years	1·202 years	1·201 years
$r=6$ to 16	1·179 years	1·186 years	1·193 years

On observational grounds, series 52 appear more likely to be homogeneous than the others. Series 50 contain an interval for which the published results are not yet definitive, and both series 50 and 51 cover intervals before and after 1922·7 when a major change in the method of reducing the observations was made. This view is supported by the fact that the estimates of the period made from series 52 show markedly less variation than the others. We therefore consider the best estimate of the period to be

$$1{\cdot}193 \text{ years} = 435{\cdot}8 \text{ mean solar days.}$$

Unfortunately there does not seem to be any acceptable way of estimating κ. As far as freedom from bias is concerned, the most reliable estimate would be obtained from formula (4a), choosing r_2 to be sufficiently large to eliminate bias due to the auto-correlation between the disturbances $\{\epsilon, \eta\}$, and $r_1 - r_2$ sufficiently large to eliminate the bias due to ignoring part of the systematic variation. On the other hand, to keep the standard error of κ^* small, r_1 and r_2 should be small, so that it becomes a question of balancing conflicting requirements. To make a theoretical estimate of the best combination of values of r_1 and r_2 is, in our opinion, impossible in view of all the unknown factors.

The results in Table X show that an oscillation persists in the estimates of κ which is such as to make it very difficult (even bearing in mind the probable

form of the bias) to decide which value to adopt. For series 52, formula (4) gives small estimated values, but it is hardly justifiable to say more than that, in the absence of bias, there is some evidence that the relaxation time lies in the range 10 to 30 years. To attach a measure of probability to this statement seems to be impossible.

20. None of the results will be acceptable if the model should be proved to be completely invalid, and it seems fair to point out that Melchior (**5**) and Gutenberg (**6**) have both expressed doubts over the question of the damping of the free motion. The case for the model was, however, very well stated by Jeffreys in (**2**); our model is a generalization of the one used by Jeffreys and is designed to take further account of observational error. As we showed in (**1**), it contains Jeffreys's model as a special case, so that we have confidence in the model at least from this aspect.

Jeffreys, however, isolated what we have called series 16 for special attention because they covered an apparently disturbance-free interval. We do not wish to add to the comments we have made on this isolation (**1**, p. 458) so far as the numerical analysis is concerned, but it should be understood that, in establishing the validity of our model, as well as that of Jeffreys, we used the assumption that there is a fairly large number of disturbances in the interval between successive observations (apart from the limiting case of completely disturbance-free motion, when its validity can be seen by putting $f_1(t)$ and $g_1(t)$ and consequently ϵ and η all equal to zero in (**1**, equations (4) and (13)). The model will certainly not be valid if there are, as Jeffreys thought, quiescent periods and highly disturbed periods, except perhaps if the observations were extended over a very large number of years indeed. This is important in the light of current geophysical theories of the Earth's core (see e.g. Revelle and Munk, **7**). If the compensations envisaged in such theories at the interface between the crust and core occur only at infrequent intervals—and such seems to be believed—then doubt must arise over the validity of our model.

21. We conclude:

(*a*) The harmonic analysis of the data is unlikely to do more than reveal qualitatively that inhomogeneity exists in the data because changes in the programme have occurred too frequently to permit the statistical theory to apply over the short sub-intervals involved.

(*b*) The unsmoothed data at monthly intervals (series of type-5) appear to be best, and of these the short series 52 are to be preferred.

(*c*) Analysis of type-5 series using high lags indicates that the estimates are biased. Qualitatively, the bias is consistent with the hypotheses that the data are inhomogeneous and that auto-correlation exists in the series of disturbances.

(*d*) The period of the free motion may be taken to be

$$1{\cdot}193 \text{ years} = 435{\cdot}8 \text{ mean solar days.}$$

(*e*) The bias is fatal for the accurate estimation of the damping factor for which, however, there is some evidence suggesting that the relaxation time is between about 10 and 30 years.

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THE THEORY OF NUTATION AND THE VARIATION OF LATITUDE

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Summary

The theory of the bodily tide and of the various nutations is developed. Elasticity of the shell and fluidity of the core are taken into account. The model used for the shell is Takeuchi's Model 2, based on one of Bullen's. The core is replaced by a homogeneous incompressible fluid, with an additional particle at the centre chosen to make the mass and moment of inertia of the core correct. Effects of the ocean are neglected, but can be allowed for at a later stage.

The period of the free nutation is found to be 392 days (which would be increased by the ocean). The bodily tide numbers for semidiurnal and long-period tides are: $h=0{\cdot}58$, $k=0{\cdot}29$, $l=0{\cdot}082$. They have other values for diurnal tides, the greatest observable differences from the statical values being for the lunar tide O. The correcting factor for the 19-yearly nutation is $0{\cdot}9964$.

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1. The treatment of periodic changes of the Earth's rotation given in most books regards the Earth as a rigid body. It has been known since Newcomb that elasticity lengthens the free period; by itself its effects on the precession and nutations are negligible. On the other hand actual fluidity of the core shortens the free period, and in the simplified case of a rigid shell and a homogeneous incompressible core it produces a reduction of the amplitude of the 19-yearly nutation that is too great to agree with the facts.

Allowance for elasticity of the shell has been found to reduce the effect of fluidity of the core on the nutation. The first treatment (H. Jeffreys, 1949) used a Wiechert–Herglotz model, with the shell and core homogeneous and incompressible. With this no more than a rough agreement was to be expected. Recent work on the velocities of elastic waves and on the density distribution has made a detailed discussion by numerical integration possible, and such a study has been carried out by H. Takeuchi (1950). His solutions are based on two of K. E. Bullen's models (1936, 1940) but the models used actually differ in some minor respects from Bullen's. Three simultaneous differential equations of the second order are solved by a Taylor series method, and the calculated values for the bodily tide numbers h, k, l are in good agreement with observation.

Takeuchi used a statical theory for both shell and core. It is known that this is valid for the shell for disturbances with the periods in question, and for the core for the semidiurnal, fortnightly, and semiannual tides. For disturbances tending to alter the axis of rotation, and therefore for all diurnal tides and the variation of latitude, a hydrodynamical theory of the core is necessary. His work, however, greatly assists such a treatment because it gives the general statical theory of the shell in terms of six adjustable constants; thus the only change needed for the shell is in the expression of the boundary conditions.

In a first attempt to use his solution the core was still treated as homogeneous and incompressible, but this was found to be unsatisfactory (H. Jeffreys, 1951b).

In the present paper, accordingly, this assumption is modified. The actual core is too complicated for detailed treatment, and simplified models are used.

We use a Lagrangian specification as before. The axis of z is in a fixed direction, while those of x and y rotate about it with a constant angular velocity ω. If x_i are the coordinates of a particle in its mean position, ξ_i in its actual position, where we shall exchange x_1, x_2, x_3, with x, y, z, and ξ_1, ξ_2, ξ_3, with ξ, η, ζ as convenient, the kinetic energy will be

$$\tfrac{1}{2}\iiint \rho_0\{(\dot{\xi}-\omega\eta)^2+(\dot{\eta}+\omega\xi)^2+\dot{\zeta}^2\}d\tau, \tag{1}$$

where ρ_0 is the undisturbed density and the integral is through the undisturbed position. There is actually a second-order variation of the angular velocity about z, but its reaction on the displacements in x and y will be of the third order.

The possibility of adapting the statical solution for the shell depends on the fact that if the displacements arose from a pure rotation $\xi\dot{\eta}-\dot{\xi}\eta$ would be of the form $\zeta\,(\xi\dot{m}-\eta\dot{l})$ and the terms in ω and ω^2 could be represented by an addition to the potential function. This requires care, however, since ξ and η are not small and second-order terms in $\dot{\xi}$, $\dot{\eta}$ need attention. We suppose the particle at x_i displaced to x_i+u_i', and then to be carried to ξ_i by a rotation, so that

$$\left.\begin{aligned}\xi_1&=x_1(1-\tfrac{1}{2}l^2)+u_1'+l(x_3+u_3')-\tfrac{1}{2}lmx_2,\\ \xi_2&=x_2(1-\tfrac{1}{2}m^2)+u_2'+m(x_3+u_3')-\tfrac{1}{2}lmx_1,\\ \xi_3&=x_3(1-\tfrac{1}{2}l^2-\tfrac{1}{2}m^2)+u_3'-l(x_1+u_1')-m(x_2+u_2').\end{aligned}\right\} \tag{2}$$

The ω^2 terms are equivalent to an addition $\tfrac{1}{2}\omega^2(\xi^2+\eta^2)$ to the potential function, and can therefore be included in the work function W. To develop this we first consider a displacement u_i. The gravitational potential consists of the undisturbed part U_0, a small external disturbance U_1, and a part U_2 due to displacements of the body. Then in a small variation of u_i the work done by gravity is

$$\delta W_g=\iiint \rho_0\frac{\partial}{\partial x_1}(U_0+U_1+U_2)\delta u_i d\tau, \tag{3}$$

where the derivative must be evaluated at x_i+u_i. To the second order

$$\delta W_g=\iiint \rho_0\left(\frac{\partial U_0}{\partial x_i}+u_k\frac{\partial^2 U_0}{\partial x_i\partial x_k}+\frac{\partial U_1}{\partial x_i}+\frac{\partial U_2}{\partial x_i}\right)\delta u_i d\tau, \tag{4}$$

all derivatives being now taken at x_i.

Since $u_i\,\partial U_2/\partial x_i$ is quadratic in the displacements this gives

$$W_g=\iiint \rho_0\left(u_i\frac{\partial U_0}{\partial x_i}+\tfrac{1}{2}u_iu_k\frac{\partial U_0}{\partial x_k}+u_i\frac{\partial U_1}{\partial x_k}+\tfrac{1}{2}u_i\frac{\partial U_2}{\partial x_i}\right)d\tau. \tag{5}$$

The stress is

$$-p_0\delta_{ik}+\lambda\Delta\delta_{ik}+2\mu e_{ik}, \tag{6}$$

where p_0 is the initial pressure at x_i,

$$\Delta=\frac{\partial u_m}{\partial x_m},\quad 2e_{ik}=\frac{\partial u_i}{\partial x_k}+\frac{\partial u_k}{\partial x_i}. \tag{7}$$

The relative increase of volume of an element is, to the second order,

$$\begin{aligned}\|\partial(x_i+u_i)/\partial x_k\|&=\frac{\partial u_i}{\partial x_i}+\frac{\partial u_2}{\partial x_2}\frac{\partial u_3}{\partial x_3}-\frac{\partial u_2}{\partial x_3}\frac{\partial u_3}{\partial x_2}+\dots\\ &=\frac{\partial u_i}{\partial x_i}+\tfrac{1}{2}\frac{\partial u_i}{\partial x_i}\frac{\partial u_k}{\partial x_k}-\tfrac{1}{2}\frac{\partial u_i}{\partial x_k}\frac{\partial u_k}{\partial x_i}.\end{aligned} \tag{8}$$

Thus p_0 does work in the expansion equal to

$$W_p = \iiint p_0\left(\frac{\partial u_i}{\partial x_i} + \tfrac{1}{2}\frac{\partial u_i}{\partial x_i}\frac{\partial u_k}{\partial x_k} - \tfrac{1}{2}\frac{\partial u_k}{\partial x_i}\frac{\partial u_i}{\partial x_k}\right)d\tau$$

$$= \iint p_0\left(l_i u_i + \tfrac{1}{2}l_i u_i\frac{\partial u_k}{\partial x_k} - \tfrac{1}{2}l_k u_i\frac{\partial u_k}{\partial x_i}\right)dS$$

$$-\iiint\left\{u_i\frac{\partial p_0}{\partial x_i} + \tfrac{1}{2}u_i\frac{\partial}{\partial x_i}\left(p_0\frac{\partial u_k}{\partial x_k}\right) - \tfrac{1}{2}u_i\frac{\partial}{\partial x_k}\left(p_0\frac{\partial u_k}{\partial x_i}\right)\right\}d\tau. \qquad (9)$$

The surface integral is zero. The part over the outer surface vanishes because $p_0 = 0$. Over the inner boundary p_0 is constant and is continuous. To the first order the total volume of matter carried through dS is $l_i u_i dS$. The second order terms might differ according as they are evaluated just inside or just outside the core. Consider the displacement imposed gradually; then u_i is the displacement of the particle that starts at x_i, while the particle that finishes at x_i starts, to the first order, at x_i' such that

$$x_i = x_i' + u_i(x_i') = x_i' + u_i + (x' - x)_k\frac{\partial u_i}{\partial x_k} \qquad (10)$$

and its displacement is $u_i - u_k\partial u_i/\partial x_k$. Thus the mean displacement of the particles that cross dS is $u_i - \tfrac{1}{2}u_k\partial u_i/\partial x_k$. With an interchange of suffixes this gives the first and third terms of the surface integral. The second term gives the increase of volume due to expansion of the matter transferred. Hence the integral is zero, since, apart from p_0, it represents the difference of the calculated volumes within the core boundary according as the calculation is based on core or shell values of the displacements, and we are supposing that the core and shell continue to fit everywhere.

Since the initial stress is hydrostatic,

$$\frac{\partial p_0}{\partial x_i} = \rho_0\frac{\partial U_0}{\partial x_i}, \qquad (11)$$

and the first order terms in W_g and W_p cancel. The second order terms give, in all,

$$W = \iiint \rho_0\left(\tfrac{1}{2}u_i u_k\frac{\partial^2 U_0}{\partial x_i\partial x_k} + u_i\frac{\partial U_1}{\partial x_i} + \tfrac{1}{2}u_i\frac{\partial U_2}{\partial x_i} - \tfrac{1}{2}u_i\frac{\partial U_0}{\partial x_i}\frac{\partial u_k}{\partial x_k} + \tfrac{1}{2}u_i\frac{\partial U_0}{\partial x_k}\frac{\partial u_k}{\partial x_i}\right)d\tau$$

$$-\tfrac{1}{2}\iiint(\lambda\Delta\delta_{ik} + 2\mu e_{ik})\frac{\partial u_i}{\partial x_k}d\tau. \qquad (12)$$

If we make small variations δu_i, apply Green's lemma, and make some interchanges of dummy suffixes, we get the equations of equilibrium

$$\frac{\partial}{\partial x_k}(\lambda\Delta\delta_{ik} + 2\mu e_{ik}) + \rho_0\frac{\partial}{\partial x_i}(U_1 + U_2) + \rho_0\left(u_k\frac{\partial^2 U_0}{\partial x_i\partial x_k} - \frac{\partial U_0}{\partial x_i}\frac{\partial u_k}{\partial x_k} + \frac{\partial U_0}{\partial x_k}\frac{\partial u_k}{\partial x_i}\right) = 0. \qquad (13)$$

The terms containing U_0 are expressible as

$$-\frac{\partial}{\partial x_k}(\rho_0 u_k)\frac{\partial U_0}{\partial x_i} + \frac{\partial}{\partial x_i}\left(\rho_0 u_k\frac{\partial U_0}{\partial x_k}\right), \qquad (14)$$

and when we use the relation

$$\frac{\partial(\rho_0, U_0)}{\partial(x_i, x_k)} = 0 \qquad (15)$$

these reduce to the terms depending on X_0, Y_0, Z_0 in the equations (esp. (20)) previously found by the Eulerian method (Jeffreys, 1929, p. 163).

We take first $u_i = u_i'$. In calculating W for u_i' we must replace U_0 by

$$\Psi = U_0 + \tfrac{1}{2}\omega^2(x_1^2 + x_2^2). \tag{16}$$

We then impose a rotation as in (2). In this the internal reactions depending on the mutual attractions and internal stresses do no work and do not alter W. The initial pressure, as modified by the inclusion of the rotation terms in Ψ, still does no work, but the extra term added to U_0 does some. The contribution is the difference between $\iiint \rho_0 \frac{1}{2}\omega^2(\xi_1^2 + \xi_2^2)d\tau$ in the positions specified by $x_i + u_i'$ and ξ_i and hence is

$$\sum_{i=1,2} \tfrac{1}{2}\omega^2 \iiint \rho(\xi_i - x_i - u_i')(\xi_i + x_i + u_i')d\tau. \tag{17}$$

Evaluating this to the second order and remembering that the axes are principal axes of the body in its standard position, we find that it is

$$-\tfrac{1}{2}\omega^2(C-A)(l^2+m^2) + \iiint \rho_0\omega^2 u_i' \frac{\partial}{\partial x_i} x_3(lx_1 + mx_2)d\tau. \tag{18}$$

The whole displacement has been separated into the displacement u_i' followed by a rotation, and so far it has been arbitrary how much of it is interpreted as belonging to u_i'. We can remove the ambiguity by introducing the condition that u_i' in the shell makes no contribution to the angular momenta; this is equivalent, to the first order, to

$$\iiint_{\text{shell}} \rho_0 \epsilon_{ikm} u_k' x_m d\tau = 0. \tag{19}$$

Dropping exact derivatives with regard to the time we have for the terms in $\xi\dot{\eta} - \dot{\xi}\eta$

$$\begin{aligned}\omega \iiint \rho_0\{(u_1' + lx_3)(\dot{u}_2' + \dot{m}x_3) - (u_2' + mx_3)(\dot{u}_1' + \dot{l}x_3)\}d\tau \\ = \omega \iiint \rho_0(l\dot{m} - \dot{l}m)x_3^2 d\tau + \omega \iiint \rho_0\{(u_1'\dot{u}_2' - u_2'\dot{u}_1') + x_3(l\dot{u}_2' - m\dot{u}_1' \\ + \dot{m}u_1' - \dot{l}u_2')\}d\tau. \end{aligned} \tag{20}$$

We can replace $l\dot{u}_2' - m\dot{u}_1'$ by $-\dot{l}u_2' + \dot{m}u_1'$, and then by (19) we can further replace the resulting integral in the shell by

$$\iiint \rho_0(-\dot{l}x_2u_3' + \dot{m}x_1u_3')d\tau. \tag{21}$$

Then (20) is replaced by

$$\begin{aligned}\omega(A - \tfrac{1}{2}C)(l\dot{m} - \dot{l}m) + \omega \iiint \rho_0(u_1'\dot{u}_2' - u_2'\dot{u}_1')d\tau \\ + \omega \iiint_{\text{shell}} \rho_0 u_i' \frac{\partial}{\partial x_i}\{x_3(\dot{m}x_1 - \dot{l}x_2)\}d\tau + 2\omega \iiint_{\text{core}} \rho_0 x_3(\dot{m}u_1' - \dot{l}u_2')d\tau. \end{aligned} \tag{22}$$

The transformation for the shell makes the second integral equal to that due to an addition to the gravitation potential. If we extend this integral to the whole Earth, the core integral is replaced by

$$\omega \iiint_{\text{core}} \rho_0\{x_3(\dot{m}u_1' - \dot{l}u_2') - u_3'(\dot{m}x_1 - \dot{l}x_2)\}d\tau. \tag{23}$$

The integrals through the whole Earth containing $\omega\dot{u}_i$ and $\omega^2 u_i$ can be taken into account by replacing U_1 by

$$U_1' = U_1 + \omega^2 x_3(lx_1 + mx_2) + \omega x_3(\dot{m}x_1 - \dot{l}x_2). \tag{24}$$

We can write to the first order

$$u_i = u_i' + u_{\omega i}, \tag{25}$$

$$u_{\omega i} = (lx_3, mx_3, -lx_1 - mx_2). \tag{26}$$

Then

$$\tfrac{1}{2}\iiint \rho_0(\dot{\xi}^2 + \dot{\eta}^2 + \dot{\zeta}^2)d\tau = \tfrac{1}{2}\iiint \rho_0(\dot{l}^2 + \dot{m}^2)(x_1^2 + x_3^2)d\tau + \iiint \rho_0(\dot{u}_i'\dot{u}_{\omega i} + \tfrac{1}{2}\dot{u}_i'^2)d\tau. \tag{27}$$

The first term is $\frac{1}{2}A(\dot{l}^2 + \dot{m}^2)$.

If

$$U_1 = x_3(c_1x_1 + c_2x_2), \tag{28}$$

$$\iiint \rho_0 u_{\omega i}\frac{\partial U_1}{\partial x_i}d\tau = -(C-A)(lc_1 + mc_2). \tag{29}$$

Then in all (integrals being through the whole Earth unless the contrary is stated)

$$\begin{aligned} L = {} & \tfrac{1}{2}A(\dot{l}^2 + \dot{m}^2) + \omega(A - \tfrac{1}{2}C)(l\dot{m} - \dot{l}m) - \tfrac{1}{2}\omega^2(C-A)(l^2 + m^2) - (C-A)(lc_1 + mc_2) \\ & + \iiint \rho_0(\dot{u}_i'\dot{u}_{\omega i} + \tfrac{1}{2}\dot{u}_i'^2)d\tau + \iiint \rho_0\omega(u_1'\dot{u}_2' - u_2'\dot{u}_1')d\tau \\ & + \iiint \rho_0 u_i'\frac{\partial U_1'}{\partial x_i}d\tau \\ & + \iiint_{\text{core}} \rho_0\omega\{x_3(\dot{m}u_1' - \dot{l}u_2') - u_3'(\dot{m}x_1 - \dot{l}x_2)\}d\tau - \tfrac{1}{2}\iiint(\lambda\Delta\delta_{ik} + 2\mu e_{ik})\frac{\partial u_i}{\partial x_k}d\tau \\ & + \iiint \rho_0 u_i'\left(\tfrac{1}{2}u_k'\frac{\partial^2\Psi}{\partial x_i\partial x_k} + \tfrac{1}{2}\frac{\partial U_2}{\partial x_i} - \tfrac{1}{2}\frac{\partial\Psi}{\partial x_i}\frac{\partial u_k'}{\partial x_k} + \tfrac{1}{2}\frac{\partial\Psi}{\partial x_k}\frac{\partial u_k'}{\partial x_i}\right)d\tau. \end{aligned} \tag{30}$$

The choice of axes and angular velocities in problems of deformable bodies has been discussed often (see especially Tisserand, 1891). The usual choice is to make the axes rotate so that the actual angular momentum at any moment agrees with that of a rigid body with the same inertia tensor as the actual body and rigidly attached to the axes. If the axes are initially principal axes, they do not in general remain so throughout the motion.

We have defined l, m so that they give the actual angular momentum of the shell, not of the whole body. It is this choice that reduces the terms in $u_1'\dot{u}_{\omega i}$ to the work done by a potential as in (22), and this is part of the standard treatment of problems of elastic rotating bodies. Its convenience is that the u_i' motion can be expressed in terms of the normal coordinates of the shell vibrating by itself. The longest period of free vibration is about 40 minutes, while we are concerned with periods of about a day or longer. Hence $\dot{u}_i'^2$ can be neglected in the shell in comparison with the elastic terms in u_i'. The same applies to $\omega(u_1'\dot{u}_2' - u_2'\dot{u}_1')$. The term in $\dot{u}_i'\dot{u}_{\omega i}$ vanishes in consequence of (19).

This is not true for the core. It is obvious that if the core boundary was exactly spherical, rotations of the shell would not affect the core at all, so that $u_i' = -u_{\omega i}$; and this is in fact approximately true for slow speeds. But the

slowest gravitational oscillation of a fluid sphere with the density of the core would have a period of about 1 hour (Lamb, 1932, p. 457). A longitudinal wave passes through the core in about 10 minutes. Hence in the core we can neglect the terms in $\dot{u}_i'^2$ that arise from compression and from motions normal to the level surfaces. We cannot however neglect those for motions along the level surfaces, since for them the free periods are very long.

2. *Lemmas on small oscillations.*—The following principles are often used in the theory.

2.1. *Quasi-statical coordinates.*—If the Lagrangian function is given by

$$2L = a_{rs}\dot{q}_r\dot{q}_s + e_{rs}q_r\dot{q}_s - b_{rs}q_rq_s + 2c_rq_r \quad (r, s = 1, 2 \ldots n) \tag{1}$$

where $\dot{q}_r$ does not appear in L for $r = k+1$ to n, we write

$$L' = L - \tfrac{1}{2}\sum_{s=k+1}^{n} q_s \frac{\partial L}{\partial q_s}. \tag{2}$$

Lagrange's equations give

$$\frac{\partial L}{\partial q_s} = 0 \quad s = k+1, \ldots n, \tag{3}$$

which determine the q_s ($s = k+1$ to n) as functions of the q_r and $\dot{q}_r(r = 1$ to $k)$. Consider small variations of the $q_r(r = 1$ to $k)$, which we shall now denote by q_i, i running from 1 to k and j from $k+1$ to n. Then if (3) still hold for the varied q_j we have for all variations of q_i

$$\frac{\partial L}{\partial q_j} = 0, \quad \delta\frac{\partial L}{\partial q_j} = 0, \quad \delta L' = \delta L. \tag{4}$$

Hence if we use (3) to eliminate q_j from L', L' can be used as the Lagrangian for the q_i. This result is exact. In the formation of L' terms quadratic in the q_j disappear and linear terms in q_j are halved. The result will ordinarily be used as an approximation when some of the free periods are very short compared with that of the motion being considered. It would, for instance, provide an easy way of estimating the correction of pendulum observations for elasticity of the rod (Jeffreys, 1956).

2.2. *Exact derivatives with regard to the time.*—We often make use of the following principle. Dr T. J. I'A. Bromwich mentioned it in lectures, and it is probably well known, but we have not found it in print after a good deal of search. If L contains a batch of terms expressible as

$$\frac{d}{dt}f(q_1, \ldots, q_n, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q_r}\dot{q}_r,$$

the contribution to Lagrange's equation for q_s is

$$\frac{d}{dt}\left(\frac{\partial f}{\partial \dot{q}_s}\right) - \frac{\partial^2 f}{\partial q_s \partial t} - \frac{\partial^2 f}{\partial q_r \partial q_s}\dot{q}_r,$$

all terms of which are immediately seen to cancel. Thus such a set of terms may be omitted from the Lagrangian.

3. Now consider small variations of u_i' in the shell. We write

$$U_2 = U_s + U_c \tag{1}$$

corresponding to contributions due to u_i' in the shell and core respectively. Then an alteration of u_i' in the shell alone does not alter U_c, but by reciprocity

$$\iiint \rho_0 u_i' \frac{\partial U_2}{\partial x_i} d\tau = \iiint_{\text{shell}} \rho_0 u_i' \frac{\partial U_s}{\partial x_i} d\tau + \iiint_{\text{core}} \rho_0 u_i' \frac{\partial U}{\partial x_i} d\tau + 2 \iiint_{\text{shell}} \rho_0 u_i' \frac{\partial U_c}{\partial x_i} d\tau. \quad (2)$$

The first integral is quadratic in shell displacements. We also put

$$U_1' = U_1 + \omega x_3 \{(\dot{m} + \omega l) x_1 - (\dot{l} - \omega m) x_2\} = x_3 (c_1' x_1 + c_2' x_2). \quad (3)$$

Then, all integrals being through the shell,

$$\delta L = \iiint \rho_0 \delta u_i' \left(\frac{\partial U_1'}{\partial x_i} + \frac{\partial U_s}{\partial x_i} + \frac{\partial U_c}{\partial x_i} \right) d\tau - \iiint (\lambda \Delta \delta_{ik} + 2\mu e_{ik}) \frac{\partial}{\partial x_k} \delta u_i' d\tau$$
$$+ \iiint \rho_0 \left\{ u_k' \frac{\partial^2 \Psi}{\partial x_i \partial x_k} \delta u_i' - \tfrac{1}{2} \frac{\partial \Psi}{\partial x_i} \frac{\partial u_k'}{\partial x_k} \delta u_i' - \tfrac{1}{2} \frac{\partial \Psi}{\partial x_k} u_i' \frac{\partial}{\partial x_k} \delta u_k' \right.$$
$$\left. + \tfrac{1}{2} \frac{\partial \Psi}{\partial x_k} \frac{\partial u_k'}{\partial x_i} \delta u_i' + \tfrac{1}{2} u_i \frac{\partial \Psi}{\partial x_k} \frac{\partial}{\partial x_i} \delta u_k' \right\} d\tau. \quad (4)$$

This gives, by application of Green's lemma and some interchanges of dummy suffixes, the equations of equilibrium (1(13)), with U_0 replaced by Ψ and U_1 by U_1'.

Now suppose that u_i' and its variations are restricted to satisfy the equations of equilibrium. Multiply these by $\frac{1}{2} u_i'$, subtract from L, and apply Green's lemma. Then the contribution of u_i' in the shell becomes

$$\tfrac{1}{2} \iiint \rho_0 u_i' \frac{\partial}{\partial x_i} (U_1' + U_c) d\tau - \tfrac{1}{2} \iint l_k u_i' (\lambda \Delta \delta_{ik} + 2 \mu e_{ik}') dS. \quad (5)$$

The halving of the terms in U_1' and U_c is an instance of the general theorem of 2.1. The surface integral depends only on surface values of u_i' and its derivatives. It represents work done by the stresses over the boundaries and can be simplified by means of the condition that there is no tangential stress.

The core integrals in lines 3, 5 and the second in line 4 of 1(30) can also be simplified. The part arising from U_s has already been included in the shell integrals. Green's lemma gives

$$\iint l_i \rho_0 u_i' \{ U_1' + \tfrac{1}{2} u_k' \frac{\partial \Psi}{\partial x_k} + \tfrac{1}{2} U_c \} dS$$
$$- \iiint \left\{ U_1' + \tfrac{1}{2} u_k' \frac{\partial \Psi}{\partial x_k} + \tfrac{1}{2} U_c \right\} \left(\rho_0 \frac{\partial u_i'}{\partial x_i} + u_i' \frac{\partial \rho_0}{\partial x_i} \right) d\tau - \tfrac{1}{2} \iiint \rho_0 u_i' \frac{\partial \Psi}{\partial x_i} \frac{\partial u_k'}{\partial x_k} d\tau \quad (6)$$

the surface integral being over the core boundary. We write u_n' for the normal displacement; and if gravity at the core boundary is g_1,

$$\frac{\partial \Psi}{\partial x_k} = - l_k g_1, \quad (7)$$

whence the surface integral is

$$\iint \rho_0 u_n' \{ U_1' - \tfrac{1}{2} g_1 u_n' + \tfrac{1}{2} U_c \} dS. \quad (8)$$

The volume integral would vanish for a uniform incompressible core and can be treated as a correction.

The normal to a boundary is as usual always taken outward from the region considered, so that at the core boundary l_i has opposite signs according as we are

considering a shell or a core integral. With these simplifications $2L$ is reduced to

$$\begin{aligned}2L &= A(\dot{l}^2+\dot{m}^2)+\omega(2A-C)(l\dot{m}-\dot{l}m)-\omega^2(C-A)(l^2+m^2)-2(C-A)(lc_1+mc_2)\\ &+\iiint_{\text{core}}\rho_0(\dot{u}_i'^2+2\dot{u}_i'\dot{u}_{\omega i})d\tau+2\iiint_{\text{core}}\rho_0\omega(u_1'\dot{u}_2'-u_2'\dot{u}_1')d\tau\\ &+2\iiint_{\text{core}}\rho_0\omega\{x_3(\dot{m}u_1'-\dot{l}u_2')-u_3'(\dot{m}x_1-\dot{l}x_2)\}d\tau\\ &+\iiint_{\text{shell}}\rho_0u_i'\frac{\partial}{\partial x_i}(U_1'+U_c)d\tau-\iint_{\text{shell}}l_ku_i'(\lambda\Delta\delta_{ik}+2\mu e_{ik})dS\\ &+\iint_{\text{core}}\rho_0u_n'(2U_1'-g_1u_n'+U_c)dS\\ &-\iiint_{\text{core}}\left(2U_1'+u_k\frac{\partial\Psi}{\partial x_k}+U_c\right)\left(\rho_0\frac{\partial u_i'}{\partial x_i}+u_i'\frac{\partial\rho_0}{\partial x_i}\right)d\tau\\ &-\iiint_{\text{core}}\rho_0u_i'\frac{\partial\Psi}{\partial x_i}\frac{\partial u_k'}{\partial x_k}d\tau-\iiint_{\text{core}}\lambda\Delta^2d\tau.\end{aligned}\tag{9}$$

4. *Form of solution for the shell.*—We take briefly

$$U_1=cK_2=cr^2S_2,\tag{1}$$

where c is a constant and K_2 a solid harmonic of degree 2. At present we ignore the difference between U_1 and U_1', which can be adjusted later when required. We also take

$$U_1+U_2=KcK_2.\tag{2}$$

K is a function of r and is a number. We introduce the variable

$$\xi=r/a,\tag{3}$$

where a is the outer radius, and write

$$x_iu_i'=q=q(\xi)K_2.\tag{4}$$

$q(\xi)$ is a number. $K'(\xi)$ is discontinuous at the boundaries, and we have

$$U_2=\{K(\xi)-1\}cK_2\quad \xi<1\tag{5}$$

$$=\{K(1)-1\}(a/r)^5cK_2\quad \xi>1,\tag{6}$$

and at the outer surface

$$\left[\frac{\partial U_2}{\partial r}\right]_{1-}^{1+}=-4\pi f\rho_0(1)\frac{q(1)}{a}K_2.\tag{7}$$

This leads to, at $\xi=1-$,

$$c\left[\frac{\partial K}{\partial\xi}+5\{K(1)-1\}\right]=4\pi f\rho_0(1)q(1).\tag{8}$$

We can also take

$$U_c=P\left(\frac{a\alpha}{r}\right)^5cK_2\quad \xi>\alpha\tag{9}$$

$$U_s=QcK_2\quad \xi<\alpha.\tag{10}$$

Then we have

$$\left(\frac{\partial U_c}{\partial r}\right)_{\alpha+}=\frac{-3}{a\alpha}PcK_2;\ \left(\frac{\partial U_c}{\partial r}\right)_{\alpha-}=\left(\frac{\partial U_c}{\partial r}\right)_{\alpha+}+4\pi f\rho_0(\alpha-)\frac{q(\alpha)}{a\alpha}K_2;\tag{11}$$

$$\left(\frac{\partial U_s}{\partial r}\right)_{\alpha-}=\frac{2}{a\alpha}QcK_2;\ \left(\frac{\partial U_s}{\partial r}\right)_{\alpha+}=\left(\frac{\partial U_s}{\partial r}\right)_{\alpha-}+4\pi f\rho_0(\alpha+)\frac{q(\alpha)}{a\alpha}K_2 \tag{12}$$

and by comparing the values of U_2 and $\partial U_2/\partial r$ at $\xi=\alpha+$ we have a pair of equations for P and Q, the solution of which is

$$5Pc=-\alpha cK'(\alpha+)+4\pi f\rho_0(\alpha+)q(\alpha) \tag{13}$$

$$5Qc=\alpha cK'(\alpha+)+5\{K(\alpha)-1\}c-4\pi f\rho_0(\alpha+)q(\alpha). \tag{14}$$

Also

$$\alpha c\left[\frac{\partial K}{\partial \xi}\right]_{\alpha-}^{\alpha+}=4\pi fq(\alpha)\{\rho_0(\alpha+)-\rho_0(\alpha-)\}. \tag{15}$$

For the displacements we take

$$u_i'=F(\xi)\frac{\partial K_2}{\partial x_i}+\frac{G(\xi)}{a^2}x_iK_2. \tag{16}$$

This is of the form of the statical solution and has the property of orthogonality with rotations used in Section 2. F and G are numbers. Let accents denote derivatives with regard to ξ. Then

$$q(\xi)=2F(\xi)+\xi^2G(\xi). \tag{17}$$

Again, with $l_k=x_k/r$, we have

$$l_k(\lambda\Delta'\delta_{ik}+2\mu e'_{ik})=\frac{\mu}{a\xi}\frac{\partial K_2}{\partial x_i}(2F+F'\xi+G\xi^2)$$
$$+\frac{x_iK_2}{a^3\xi^2}[\lambda(2F'+\xi^2G'+5\xi G)+\mu(2F'+2\xi^2G'+4\xi G)]. \tag{18}$$

Since there is no tangential stress at the boundaries

$$\xi F'+2F+\xi^2G=0,\quad \xi=1,\alpha \tag{19}$$

and the elasticity contribution to $2L$ is

$$-a^3\xi^4M[\lambda(2F'+\xi^2G'+5\xi G)+\mu(2F'+2\xi^2G'+4\xi G)]q(\xi), \tag{20}$$

where

$$M=\iint S_2^2d\Omega=\tfrac{4}{15}\pi \tag{21}$$

when K_2 is the solid harmonic xz or yz.
For $\xi=1$, (18) is correct. For $\xi=\alpha$, it must be taken with the opposite sign.

For integration over a sphere we have the identity

$$\iint\left(\frac{\partial K_n}{\partial x_i}\right)^2dS=\frac{n(2n+1)}{r^2}\iint K_n^2dS. \tag{22}$$

Now

$$U_1+U_c=c\{1+P(\alpha/\xi)^5\}K_2, \tag{23}$$

whence the volume integral in $2L$ is

$$Ma^5c\int_\alpha^1\rho_0\left(10F+2\xi^2G-3GP\frac{\alpha^5}{\xi^3}\right)\xi^4d\xi. \tag{24}$$

(18) is already expressed in terms of boundary values of F, G and their first

derivatives. (24) still contains internal values of F and G, and the corresponding expression requires a numerical integration. Takeuchi solves the differential equations for F, G and K (the last being Poisson's equation) in terms of their values and their first derivatives at $\xi = \alpha +$. His functions, however, are multiples of those used here; in particular he takes $c = 1$. In view of the pair of boundary conditions (19) we can eliminate two of the constants; and on account of the importance of $q(1)$ and $q(\alpha)$ it is convenient to eliminate two others in favour of them. Further, c is a datum, and we have the relation (8) connecting $K'(1)$ and $K(1)$. One constant remains arbitrary, and on account of the prominence of P it seems best to take $K'(\alpha +)$, which is wholly determined by the displacements of the core. When this is done the internal values of u_i' in the shell will have been eliminated.

If we indicate Takeuchi's functions by the suffix T, the correspondence between his and ours is

$$\left.\begin{aligned} F(\xi) &= ca^2F_T(\xi); & G(\xi) &= ca^2G_T(\xi); \\ q(\xi) &= ca^2(2F_T+G_T); & K(\xi) &= 4\pi f a^2 K_T(\xi). \end{aligned}\right\} \tag{25}$$

Since his equations are all homogeneous the factor a^2 can be omitted. We write $c/4\pi f = D$ and take the correspondence to be

F	G	q	DK
F_T	G_T	q_T	K_T

D and ρ are densities, and where they appear in the numerical solution they must be interpreted as $(D, \rho)/(1\ \mathrm{g/cm^3})$.

Then (8) can be written

$$D\{K'(1) + 5K(1)\} = 5D + \rho(1)q(1). \tag{26}$$

Also

$$2F(1) + G(1) = q(1), \tag{27}$$

$$F'(1) = -q(1), \tag{28}$$

and with $\alpha = 0{\cdot}5447$,

$$2F(\alpha) + 0{\cdot}2967\ G(\alpha) = q(\alpha), \tag{29}$$

$$0{\cdot}5447\ F'(\alpha) = -q(\alpha), \tag{30}$$

$$DK'(\alpha) = cK_T'(\alpha). \tag{31}$$

Dr Takeuchi very kindly gave us a copy of his matrices for his Model 2 connecting $F(\xi)$, $G'(\xi)$ ... $K(\xi)$ with their values at $\xi = \alpha$, which are not given in detail in his paper. These were transformed to the four constants used here, by means of the six equations above. There are discrepancies between the last lines in his (198) and (205); the second entry should be 0·072887, the fourth 0·23256 in both cases.

The solution is as follows:

$$\left.\begin{aligned} F'(\alpha) &= -1{\cdot}8359\ q(\alpha), \\ G'(\alpha) &= 24{\cdot}6632\ q(1) - 10{\cdot}2459\ q(\alpha) - 0{\cdot}00013\ DK'(\alpha) + 1{\cdot}8392\ D, \\ DK'(\alpha) &= DK'(\alpha), \\ F(\alpha) &= 0{\cdot}31942\ q(1) + 0{\cdot}18614\ q(\alpha) - 0{\cdot}00062\ DK'(\alpha) + 0{\cdot}03546\ D, \\ G(\alpha) &= -2{\cdot}1524\ q(1) + 2{\cdot}11499\ q(\alpha) + 0{\cdot}004145\ DK'(\alpha) - 0{\cdot}23895\ D, \\ DK(\alpha) &= 0{\cdot}00078\ q(1) + 0{\cdot}3555\ q(\alpha) - 0{\cdot}1110\ DK'(\alpha) + 1{\cdot}1020\ D, \end{aligned}\right\} \tag{32}$$

$$\left.\begin{aligned} F'(1)&=-q(1),\\ G'(1)&=3\cdot50477\,q(1)-2\cdot09580\,q(\alpha)-0\cdot00325\,DK'(\alpha)-0\cdot1476\,D,\\ DK'(1)&=1\cdot08059\,q(1)-0\cdot59626\,q(\alpha)+0\cdot02873\,DK'(\alpha)-0\cdot22403\,D,\\ F(1)&=0\cdot12191\,q(1)-0\cdot06591\,q(\alpha)-0\cdot00058\,DK'(\alpha)+0\cdot04254\,D,\\ G(1)&=0\cdot75622\,q(1)+0\cdot13178\,q(\alpha)+0\cdot00118\,DK'(\alpha)-0\cdot08488\,D,\\ DK(1)&=0\cdot38367\,q(1)+0\cdot11938\,q(\alpha)-0\cdot00578\,DK'(\alpha)+1\cdot04472\,D.\end{aligned}\right\} \quad (33)$$

Some features of the solution are worth special mention. The coefficient of D in $K(\xi)$ never deviates far from 1. If motions of the core and the boundaries were prevented, the displacements of the shell would have little effect on the potential. The coefficient of $q(1)$ in $DK(\alpha)$ is very small. With a homogeneous incompressible shell and core, and with $q(\alpha)=0$ and no external field, $K(\xi)$ would be constant. This is the problem of distortion by loading over the outer surface, with a constraint preventing core motions. But here the coefficient of $q(1)$ in $K(\xi)$ varies by a factor of 50. In a uniform sphere the effects of compressibility are usually not large, but it appears that in this problem they are. The large coefficient of $q(1)$ in $G'(\alpha)$ corresponds to a large disturbance of density just outside the core, and it appears that the reduction of potential due to this cancels most of the increase due to the elevation of the outer surface. The coefficients of $q(\alpha)$ and $DK'(\alpha)$ in $DK(\xi)$ decrease greatly outward; this is mainly due to the factor ξ^{-5} in the contribution from P.

Extra figures were calculated in some of the coefficients because of cancellation in the checking.

The solutions for other values of ξ were calculated on the EDSAC machine at the Mathematical Laboratory, Cambridge, by Miss M. Lewin (now Mrs Mutch). As Takeuchi's last two figures are mainly guard figures they are rounded off at this stage, and the solutions are in Table I. The values of ξ are as follows:

$\xi_0=\alpha=0\cdot54474$, $\xi_1=0\cdot59184$, $\xi_2=0\cdot63893$, $\xi_3=0\cdot68603$, $\xi_4=0\cdot73313$, $\xi_5=0\cdot78022$, $\xi_6=0\cdot82732$, $\xi_7=0\cdot87441$, $\xi_8=0\cdot92151$, $\xi_9=0\cdot96075$, $\xi_{10}=1\cdot00000$.

The intervals for r, measured from the outer surface, are 250 km to 500 km, then 300 km to 2900 km. The data assume a discontinuous change of properties at a depth of 500 km. The rows in the matrices give F', G', DK', F, G, K for each ξ.

In the integration we need only F and G, but some of the coefficients vary too rapidly for accurate integration by the Gregory formula. Since, however, F' and G' were also available, the interval was halved by the formula (Jeffreys, 1953)

$$f(\tfrac{1}{2}h)=\tfrac{1}{2}f(0)+\tfrac{1}{2}f(h)-\tfrac{1}{8}h\{f'(h)-f'(0)\}. \quad (34)$$

This formula is more accurate than the Newton–Bessel formula with second differences and can be used at the end intervals. When this was done integration by Gregory's formula was accurate enough.

After integration it was convenient to use (13) to eliminate $DK(\alpha+)$; with the values adopted $\rho_0(\alpha+)=5\cdot53\ \mathrm{g/cm^3}$,

$$DK'(\alpha+)=-9\cdot1787\,DP+10\cdot152\,q(\alpha). \quad (35)$$

Then the surface integrals give in $2L$ (lines 4, 5 of 3(9))

$$-Ma^3\times10^{12}\,q(1)\{8\cdot415\,q(1)-3\cdot168\,q(\alpha)-0\cdot00670\,DP-0\cdot7698\,D\}, \quad (36)$$

$$-Ma^3\times10^{12}\,q(\alpha)\{-3\cdot157\,q(1)+1\cdot644\,q(\alpha)+0\cdot0631\,DP-0\cdot1047\,D\}, \quad (37)$$

and the volume integral (line 4) gives

$$Ma^3 \times 10^{12}(0{\cdot}7401\, q(1)\, D + 0{\cdot}1003\, q(\alpha) D + 0{\cdot}0164\, D^2P + 0{\cdot}0744\, D^2 + 0{\cdot}0068\, q(1)\, DP - 0{\cdot}0621\, q(\alpha)\, DP + 0{\cdot}0017\, D^2P^2. \tag{38}$$

C.g.s. units are understood.

We have two useful checks. $q(1)$ and $q(\alpha)$ are Lagrangian coordinates, and by a reciprocity theorem the coefficients of $q(1)\, q(\alpha)$ in the two surface integrals should be equal. They agree to about 1 part in 300. Also D and $D\dot{P}$ represent external forces, acting on the shell, and if $q(1)$ and $q(\alpha)$ are put zero the terms in D and DP will represent work done by the external forces with the boundaries fixed. Then the amounts of work done by D on the displacements due to DP and done by DP on those due to D should be equal, by another reciprocity theorem, and the coefficient 0·0164 is in fact the sum of two of 0·0082. These comparisons are checks both on the present work and Takeuchi's.

Table I

	$q(1)$	$q(\alpha)$	$DK'(\alpha)$	D
Y_1	−0·069	−1·296	0·0002	0·014
	15·198	−8·770	−0·009	1·056
	0·486	−0·468	0·611	−0·042
	0·317	0·115	−0·0006	0·036
	−1·238	1·663	0·0039	−0·173
	0·020	0·343	−0·074	1·101
Y_2	−0·211	−0·892	0·0003	0·014
	9·829	−6·918	−0·011	0·631
	0·723	−0·620	0·388	−0·073
	0·310	0·065	−0·0006	0·036
	−0·662	1·293	0·0034	−0·134
	0·049	0·316	−0·051	1·100
Y_3	−0·297	−0·661	0·0002	0·016
	6·613	−5·296	−0·010	0·390
	0·847	−0·634	0·255	−0·097
	0·298	0·029	−0·0006	0·037
	−0·282	1·007	0·0029	−0·111
	0·086	0·287	−0·036	1·094
Y_4	−0·371	−0·503	0·0001	0·018
	4·610	−4·054	−0·008	0·249
	0·922	−0·593	0·173	−0·117
	0·282	0·001	−0·0006	0·038
	−0·021	0·789	0·0025	−0·096
	0·128	0·258	−0·026	1·089
Y_5	−0·437	−0·394	0·0000	0·019
	3·369	−3·139	−0·007	0·161
	0·972	−0·540	0·120	−0·134
	0·263	−0·020	−0·0006	0·039
	0·163	0·621	0·0022	−0·087
	0·172	0·231	−0·019	1·083
Y_6	−0·514	−0·310	−0·0000	0·018
	2·599	−2·467	−0·005	0·099
	1·009	−0·493	0·085	−0·149
	0·240	−0·036	−0·0006	0·040
	0·302	0·490	0·0019	−0·081
	0·219	0·207	−0·015	1·076

	$q(1)$	$q(\alpha)$	$DK'(\alpha)$	D
Y_7	−0·589	−0·239	−0·0001	0·017
	2·163	−2·032	−0·005	0·051
	1·037	−0·461	0·062	−0·164
	0·214	−0·049	−0·0006	0·040
	0·413	0·385	0·0016	−0·077
	0·267	0·185	−0·011	1·069
Y_8 in	−0·667	−0·175	−0·0001	0·016
	2·105	−1·832	−0·004	0·003
	1·087	−0·454	0·046	−0·180
	0·184	−0·058	−0·0006	0·041
	0·511	0·296	0·0014	−0·076
	0·317	0·163	−0·009	1·061
Y_8 out	−0·603	−0·184	−0·0002	0·028
	2·949	−2·177	−0·004	−0·097
	0·626	−0·531	0·046	−0·190
	0·184	−0·058	−0·0006	0·041
	0·511	0·296	0·0014	−0·076
	0·317	0·163	−0·009	1·061
Y_9	−0·788	−0·094	−0·0001	0·017
	3·090	−2·072	−0·003	−0·113
	0·856	−0·560	0·036	−0·205
	0·157	−0·064	−0·0006	0·042
	0·628	0·213	0·0013	−0·080
	0·346	0·142	−0·007	1·053
Y_{10}	−1·000	−0·000	−0·0000	−0·000
	3·505	−2·096	−0·003	−0·148
	1·081	−0·596	0·029	−0·224
	0·122	−0·066	−0·0006	0·042
	0·756	0·132	0·0012	−0·085
	0·384	0·119	−0·006	1·045

5. *Models for the core.*—The actual core is too complex for detailed treatment, but useful simplifications are possible. It has appeared probable that so long as the moments of inertia of the core and the ellipticity of its boundary are correct, the detailed distribution of density is of secondary importance. But if the Radau approximation is valid, the mean moment of inertia is enough to determine the other quantities. Now Bullen (1942) found that extreme hypotheses gave for the core

$$\frac{I}{Ma^2} = 0{\cdot}389 \text{ and } 0{\cdot}373$$

in comparison with 0·400 for a homogeneous core. Now it has been shown in a former paper (Jeffreys, 1951 *a*), in which $K=\frac{5}{8}$ corresponds to $I/Ma^2=\frac{1}{3}$, that the ratio e/m for a Wiechert model varies over the whole range of possible core radii by only 2·5 per cent. For $I/Ma^2=0{\cdot}4$, the body is homogeneous and the ratio does not depend at all on the radius that we might choose to be that of the core. Thus it seems likely that for the Earth's core (in which the inner core would play the part of the core of a Wiechert model, and the potential due to the figure of the shell would be added to that due to rotation) the Radau approximation will always give solutions valid within 1 per cent.

Through most of the core the variation of density is mainly due to compression. If this held for the whole of the core a smooth distribution of Laplace's or Roche's

type would fit fairly well. It is desirable also to have an estimate of the effect of the inner core, but for simplicity we shall treat this as a point mass at the centre.

Bullard's (1946) density distribution makes the moment of inertia of the core 0·107 of that of the whole Earth, and the ellipticity 0·002567. The dynamical ellipticity is 0·002562; this was computed from his solution by numerical integration. We shall adopt these values in all models. The reason for the close agreement is that the dynamical ellipticity is not a weighted mean of the internal ellipticities in the usual sense. In computing it we compare integrals of the forms $\int \rho d(r^5 e)$ and $\int \rho dr^5$, whose ratio would be e if e was constant. With variable e, the smaller e near the centre reduces the first integral, but this is compensated by the fact that the consequent reduction of $r^5 e$ at intermediate depths has to be made up by increased values of its steps as the boundary is approached, so that the dominant part of the integrals still comes from values near the surfaces.

We consider two extreme models for the core. In the first the core is treated as homogeneous and incompressible, with a point mass added at the centre. In the second we take a Roche density law

$$\rho = k_0 - k_1 \xi_c^2 \tag{1}$$

where $a_1 = a\alpha$ is the radius of the core and $\xi_c = r/a_1$. This makes the mean density

$$\bar{\rho} = k_0 - \tfrac{3}{5} k_1, \tag{2}$$

and the moment of inertia is that of a homogeneous sphere of density

$$\rho_c = k_0 - \tfrac{5}{7} k_1. \tag{3}$$

Bullard's values for the density lead to

$$\bar{\rho} = 10{\cdot}70, \quad \rho_c = 10{\cdot}28\ \mathrm{g/cm^3}, \tag{4}$$

whence

$$k_0 = 12{\cdot}91\ \mathrm{g/cm^3}, \quad k_1 = 3{\cdot}68\ \mathrm{g/cm^3}, \quad k_1/k_0 = 0{\cdot}285. \tag{5}$$

For a distribution of density ρ_c with a central particle of mass m_1 the mean density is

$$\bar{\rho} = \rho_c + \frac{m_1}{\tfrac{4}{3}\pi a_1^3}, \tag{6}$$

whence

$$\frac{m_1}{\tfrac{4}{3}\pi \rho_c a_1^3} = 0{\cdot}041. \tag{7}$$

We note that a 28 per cent variation of density in the Roche model is equivalent to putting 4 per cent of the mass in a central particle, so far as the hydrostatic theory is concerned.

More can be said; for Radau's approximation is good for any Roche model, and it is easy to show that for any model satisfying it the surface ellipticity of a free body is given by

$$\frac{e}{\tfrac{5}{4} m} = \frac{1}{1 - \tfrac{3}{2}\left(\dfrac{5I}{2Ma^2} - 1\right)}, \tag{8}$$

to the first order in the small quantity $5I/2Ma^2 - 1$. But for the central particle model we can easily show that the same is true. Also to the same order the

dynamical and geometrical ellipticities are equal. There can therefore be no inconsistency in adopting Bullard's ellipticity of the core, namely 0·002567, in both models.

For Roche's models we take the undisturbed density over a level surface to be

$$\rho_0 = k_0 - k_1 \xi_e^2, \tag{9}$$

where $a_1\xi_e$ is now the mean radius of the surface, and the corresponding geopotential is

$$\Psi = -4\pi f a_1^2(\tfrac{1}{6}k_0\xi_e^2 - \tfrac{1}{20}k_1\xi_e^4). \tag{10}$$

By Radau's approximation the ellipticity of a stratum of uniform density is

$$e = e_1\left\{1 - \tfrac{6}{35}\frac{k_1}{k_0}(1-\xi_e^2)\right\}, \tag{11}$$

where e_1 refers to the boundary. Then the equation of the stratum is

$$r/a_1 = \xi_e\{1 + e(\tfrac{1}{3} - \cos^2\theta)\} \tag{12}$$

and we put

$$W_2 = r^2(\tfrac{1}{3} - \cos^2\theta) = \tfrac{1}{3}(x^2 + y^2 - 2z^2). \tag{13}$$

Then to orders k_1 and e_1

$$\rho = k_0 - k_1\frac{r^2}{a^2} + \frac{2k_1e_1}{a^2}W_2, \tag{14}$$

$$\Psi = -\tfrac{2}{3}\pi f\left(k_0r^2 - \tfrac{3}{10}k_1\frac{r^4}{a^2}\right) + \tfrac{4}{3}\pi f(k_0 - \tfrac{6}{35}k_1)e_1W_2 - \tfrac{4}{7}\pi f k_1 e_1\frac{r^2}{a^2}W_2. \tag{15}$$

We shall not need the ellipticities of surfaces of equal density again, and shall therefore denote the ellipticity of the core boundary simply by e.

6. *Solutions for central particle models: statical case.*—In this case the volume integrals in 3(9) disappear; and

$$U_c = \tfrac{4}{5}\pi f\rho_c q(\alpha)\left(\frac{a\alpha}{r}\right)^5 K_2, \tag{1}$$

whence

$$DP = \tfrac{1}{5}\rho_c q(\alpha) = 2{\cdot}056\,q(\alpha). \tag{2}$$

Also if the normal displacement is

$$u_n' = q(\alpha)K_2/a\alpha, \tag{3}$$

$$\begin{aligned}
&\iint l_i\rho_c u_i'\left\{2U_1 + U_c + u_k'\frac{\partial\Psi}{\partial x_k}\right\}dS \\
&= \iint \rho_c u_n'(2U_1 + U_c - l_k g_1 u_k')dS \\
&= 4\pi f a^2 . Ma^3 . \alpha^5\rho_c\left(2D - \tfrac{2}{15}\rho_c q(\alpha) - \frac{m_1}{4\pi a_1^3}q(\alpha)\right)q(\alpha) \\
&= 0{\cdot}1677\times 10^{12}Ma^3\{2Dq(\alpha) - 1{\cdot}51\,q^2(\alpha)\}.
\end{aligned} \tag{4}$$

We add this to 3(36) (37) (38) to give the potential terms in $2L$. We take first the statical case. We have in all

$$\begin{aligned}
\frac{2L}{Ma^3\times 10^{12}} = &-8{\cdot}415\,q(1)^2 + 6{\cdot}353\,q(1)q(\alpha) - 2{\cdot}147\,q(\alpha)^2 \\
&+ 1{\cdot}510\,q(1)D + 0{\cdot}574\,q(\alpha)D + 0{\cdot}0744D^2
\end{aligned} \tag{5}$$

whence

$$q(1) = 0{\cdot}317D,\quad q(\alpha) = 0{\cdot}603D,\quad K(1) = 1{\cdot}289,\quad F(1) = 0{\cdot}0444D. \tag{6}$$

Love's numbers h and k are

$$h = \frac{g}{a}\frac{q(1)}{c} = \tfrac{1}{3}\bar{\rho}\frac{q(1)}{D} = 0\cdot585 \tag{7}$$

$$k = K(1) - 1 = 0\cdot289 \tag{8}$$

and the Shida–Lambert number is

$$l = \tfrac{1}{3}\bar{\rho}\frac{F(1)}{D} = 0\cdot082. \tag{9}$$

For comparison Takeuchi's values are

	h	k	l
Model 1	0·587	0·290	0·068
Model 2	0·610	0·281	0·082

Agreement is good, but as the present model for the crust is nearer to Takeuchi's Model 2 than to Model 1, it is rather surprising that two of the numbers agree better with Model 1.

The factor M has different values for different second harmonics, but cancels in the statical case, which is applicable to the semidiurnal and long-period tides.

7. *Central particle model: dynamical case.*—When l, m are not zero $\ddot{u}_i'$ can no longer be neglected in the core. The statical solution for the shell must be modified by restoring U_1'.

u_i' can be broken into two parts,

$$u_i' = v_{i1} + v_{i2}, \tag{1}$$

where

$$v_{i1} = \frac{l_1 z a_1}{c_1},\ \frac{m_1 z a_1}{c_1},\ -\frac{(l_1 x + m_1 y)c_1}{a_1}, \tag{2}$$

$$v_{i2} = \frac{l_2 z a_1}{c_1},\ \frac{m_2 z a_1}{c_1},\ \frac{(l_2 x + m_2 y)c_1}{a_1}. \tag{3}$$

This makes u_i' linear in the coordinates; this is justified because the obstruction due to the central particle will be of no importance since $u_i' \to 0$ at the centre.

The normal displacement at the boundary is

$$u_n' = \frac{2R}{a_1 c_1} z(l_2 x + m_2 y), \tag{4}$$

where

$$\frac{1}{R^2} = \frac{x^2 + y^2}{a_1^4} + \frac{z^2}{c_1^4}. \tag{5}$$

u_n' must be continuous with the normal displacement of the shell; since this is an elastic displacement R can be replaced by a_1.

We introduce the constants

$$A_{11} = \iiint \rho_0 x^2 d\tau,\ A_{33} = \iiint \rho_0 z^2 d\tau; \tag{6}$$

$$C_1 = 2A_{11},\ A_1 = A_{11} + A_{33},\ F = 2\frac{a}{c}A_{33} = 2\frac{c}{a}A_{11}. \tag{7}$$

Then

$$F^2 = 4A_{11}A_{33} = C_1(2A_1 - C_1). \tag{8}$$

The ellipticity of the core being small, we have nearly

$$C_1 = A_1(1 + e),\ 2A_1 - C_1 = A_1(1 - e) \tag{9}$$

and hence

$$F=A_1(1+O(e^2)). \tag{10}$$

We find

$$\iiint \rho \dot{u}_i'^2 d\tau = A_1(\dot{l}_1^2+\dot{l}_2^2+\dot{m}_1^2+\dot{m}_2^2)+2(C_1-A_1)(\dot{l}_1\dot{l}_2+\dot{m}_1\dot{m}_2) \tag{11}$$

$$2\iiint \rho \dot{u}_{\omega i}\dot{u}_i' d\tau = 2F(\ddot{l}l_1+\ddot{m}m_1) \tag{12}$$

$$2\omega\iiint \rho_0(u_1'\dot{u}_2'-u_2'\dot{u}_1')d\tau=\omega C_1(l_1\dot{m}_1-m_1\dot{l}_1+l_2\dot{m}_1-m_1\dot{l}_2+l_1\dot{m}_2$$
$$-m_2\dot{l}_1+l_2\dot{m}_2-m_2\dot{l}_2)$$

which is equivalent to

$$\omega C_1(l_1\dot{m}_1-m_1\dot{l}_1+2l_2\dot{m}_1-2m_2\dot{l}_1+2l_2\dot{m}_2). \tag{13}$$

The term in $l_2\dot{m}_2$ can be dropped, since it is of the order of the kinetic energy of the elastic displacements. But those in $l_2\dot{m}_1-m_2\dot{l}_1$ are of the order of those arising from the correcting terms in U_1', and must be retained.

$$2\iiint \rho_0\omega\{x_3(\dot{m}u_1'-\dot{l}u_2')-u_3'(\dot{m}x_1-\dot{l}x_2)\}d\tau=2F\omega(\dot{m}l_1-\dot{l}m_1). \tag{14}$$

The core integrals arising from the last three lines of 3(9) are the same as in the statical case except that U_1 has to be replaced by U_1', and with this change 6(4) will hold.

We take $A_1=0\cdot107C$, $C_1=A_1=eA_1$, $e=0\cdot002562$;

$a_1/a=\alpha=0\cdot5447$; $\omega=7\cdot2921\times10^{-5}/\text{sec}$; $C=8\cdot05\times10^{44}\,\text{gm cm}^2$;

$A=C(1-H)$, $H=0\cdot003273$;

$a=6\cdot371\times10^8\,\text{cm}$.

Then

$$C\omega^2=4\cdot28\times10^{36}\,\text{gm cm}^2/\text{sec}^2;$$
$$M=\tfrac{4}{15}\pi,\quad Ma^3=2\cdot166\times10^{26}\,\text{cm}^3,$$
$$10^{12}\frac{Ma^3}{C\omega^2}=50\cdot61.$$

We keep H and e explicit for the present; this facilitates some checks. We take also

$$q(1)K_2=r_1xz+r_2yz$$
$$q(\alpha)K_2=s_1xz+s_2yz;\quad l_2=\tfrac{1}{2}s_1,\quad m_2=\tfrac{1}{2}s_2$$
$$DK_2=d_1'xz+d_2'yz$$
$$=\frac{\omega^2}{4\pi f}\left\{\left(\frac{c_1}{\omega^2}+\frac{\dot{m}}{\omega}+l\right)xz+\left(\frac{c_2}{\omega^2}-\frac{\dot{l}}{\omega}+m\right)yz\right\} \tag{15}$$

and

$$\omega^2/4\pi f=0\cdot006345.$$

It is convenient to use $2L/C\omega^2$ instead of $2L$. Then

$$\frac{2L}{C\omega^2}=-425\cdot9(r_1^2+r_2^2)+321\cdot5(r_1s_1+r_2s_2)-108\cdot7(s_1^2+s_2^2)+76\cdot4(r_1d_1'+r_2d_2')$$
$$+29\cdot05(s_1d_1'+s_2d_2')+3\cdot765(d_1'^2+d_2'^2).$$
$$+\frac{1}{\omega^2}[(1-H)(\dot{l}^2+\dot{m}^2)+\omega(1-2H)(l\dot{m}-\dot{l}m)-\omega^2H(l^2+m^2)$$
$$+0\cdot107(\dot{l}_1^2+\dot{m}_1^2)+0\cdot107\,e(\dot{l}_1\dot{s}_1+\dot{m}_1\dot{s}_2)$$
$$+0\cdot214(\ddot{l}l_1+\ddot{m}m_1)+0\cdot107(1+e)\omega(l_1\dot{m}_1-m_1\dot{l}_1+s_1\dot{m}_1-s_2\dot{l}_1)$$
$$+0\cdot214\omega(\dot{m}l_1-\dot{l}m_1)-2H(lc_1+mc_2)]. \tag{16}$$

The terms in $e\dot{l}_1 s_1$, $\omega e s_1 \dot{m}_1$ can be dropped in comparison with those that do not contain e.

We put
$$c_1 + ic_2 = k, \quad l + im = \zeta, \quad l_1 + im_1 = \zeta_1. \tag{17}$$

Derivatives of r_1, r_2, s_1, s_2 do not appear in the Lagrangian; hence these coordinates can be eliminated.

We find
$$(r_1, r_2) = 0{\cdot}3774(s_1, s_2) + 0{\cdot}0897(d_1', d_2'), \tag{18}$$
$$(s_1, s_2) = 0{\cdot}607(d_1', d_2') + \frac{0{\cdot}001115}{\omega}(\dot{m}_1, -\dot{l}_1), \tag{19}$$

and the modified Lagrangian is given by
$$\begin{aligned}\frac{2L'}{C} = {} & (1 - H + 0{\cdot}00100)(\dot{l}^2 + \dot{m}^2) + \omega(1 - 2H + 0{\cdot}00200)(l\dot{m} - \dot{l}m) \\ & - \omega^2(H - 0{\cdot}00100)(l^2 + m^2) \\ & + 0{\cdot}10706(\dot{l}_1^2 + \dot{m}_1^2) + 0{\cdot}21441(\dot{l}\dot{l}_1 + \dot{m}\dot{m}_1) + 0{\cdot}107\omega(1 + e)(l_1\dot{m}_1 - m_1\dot{l}_1) \\ & + 0{\cdot}214\omega(\dot{m}l_1 - \dot{l}m_1) + 0{\cdot}00041\omega(l\dot{m}_1 - \dot{l}_1 m) \\ & + (-2H + 0{\cdot}00200)(c_1 l + c_2 m) + \frac{0{\cdot}00200}{\omega}(c_1\dot{m} - c_2\dot{l}) \\ & + \frac{0{\cdot}00041}{\omega}(c_1\dot{m}_1 - c_2\dot{l}_1) + \frac{0{\cdot}00100}{\omega^2}(c_1^2 + c_2^2).\end{aligned} \tag{20}$$

We form the equations of motion and assume a time factor $e^{i\gamma t}$. Two speeds of free motions are found to be 0 and $-\omega$. This gives a valuable check, since the speed 0 corresponds to a statical displacement of the core without moving the shell and that of $-\omega$ gives a steady displacement of the axis in space. A further check is that for $\gamma = -\omega$ the rate of motion in space of the axis, given by $(\gamma + \omega)\zeta$, is found to be the same as for a rigid body. This represents the precession, which in all previous models has been found to be independent of the internal constitution.

For the other free motions we find
$$\frac{\gamma}{\omega} = 0{\cdot}00255; \quad \frac{\omega}{\gamma} = 392{\cdot}4; \quad \frac{\zeta_1}{\zeta} = -0{\cdot}99931, \tag{21}$$
$$\frac{\gamma + \omega}{\omega} = -0{\cdot}00224; \quad \frac{\omega}{\gamma + \omega} = -447; \quad \frac{\zeta_1}{\zeta} = -9{\cdot}33. \tag{22}$$

The first of these is the variation of latitude. The period is shorter than any found from observation. It would be lengthened by the effect of the oceans; it remains to be seen whether compressibility of the core would make much difference. There is very little motion of the core, since apart from the effect of the ellipticity the displacements due to ζ and ζ_1 nearly cancel.*

The second movement is not observed. The core displacements are opposite to the general rotation and much larger.

* A former solution (Jeffreys, 1951 *b*) by the method of undetermined multipliers gave 463 days for the period. A homogeneous core was assumed. This differs from the present model only by the central particle, and a great difference was not to be expected. The difference has been traced to a misplaced 2 in the earlier solution, which when corrected leads to 390 days.

For the forced motions we write $\gamma + \omega = n$. As a standard of comparison we take the value of ζ for a rigid Earth and call this ζ_0. We have

$$Hk = -n\zeta_0(\omega - n + nH), \tag{23}$$

and the equations of motion become

$$(\omega - 0{\cdot}99773n)\zeta + 0{\cdot}10720(\omega - n)\zeta_1 = (\omega - 0{\cdot}99673n)\left(1 - 0{\cdot}3055\frac{n}{\omega}\right)\zeta_0, \tag{24}$$

$$0{\cdot}10720n\zeta + (0{\cdot}000214\omega + 0{\cdot}10706n)\zeta_1 = 0{\cdot}0611n\left(1 - 0{\cdot}3055\frac{n}{\omega}\right)\zeta_0. \tag{25}$$

The coefficient 0·3055 is $0{\cdot}00100/H$, and is a consequence of elasticity of the shell, not of fluidity of the core.

Results for the most interesting speeds are as follows:

n/ω	ζ/ζ_0	ζ_1/ζ_0	Tidal component	Nutation component
$-\frac{1}{13{\cdot}7}$	1·0768	−0·509	OO	Secondary fortnightly
$-\frac{1}{183}$	1·0895	−0·819		
$-\frac{1}{6800}$	0·9964	0·034	K_1 (companion)	Principal nutation
0	1·0000	0·000	K_1	Precession
$\frac{1}{6800}$	1·0036	−0·034	K_1 (companion)	Principal nutation (correction)
$\frac{1}{183}$	1·0350	−0·342	P	Semiannual
$\frac{1}{13{\cdot}7}$	1·0269	−0·458	O	Fortnightly

For $n/\omega = -1/447$, ζ/ζ_0 and ζ_1/ζ_0 become infinite and change sign; and for $n/\omega = -1/479$, ζ vanishes and ζ_1 does not. This is the explanation of the peculiar behaviour between $-1/183$ and $-1/6800$.

On the whole the results are fairly similar to those found for a rigid shell. The principal difference is that for a rigid shell, unless $|n/\omega|$ is small of order e, ζ_1 is nearly $-\zeta$ and ζ/ζ_0 is about 1·12. This is not true with an elastic shell. Even for O and OO ζ_1 is only about $-0{\cdot}5\,\zeta_0$, and for the principal lunar diurnal tide O the predicted amplitude of ζ is under 3 per cent in excess of the rigid body value. H. R. Morgan (1952) has made a thorough analysis of the observations and has finally decided that they will not determine any significant correction.

For a rigid shell the correcting factor to the principal 19-yearly nutation was 0·9940. The result here is 0·9964. For comparison, Newcomb's standard value of the nutation in obliquity is $9''{\cdot}210$. The rigid body value based on Spencer Jones's values of the solar parallax and the lunar inequality is $9''{\cdot}2272 \pm 0''{\cdot}0008$ (p.e.) Rabe's parallax and my value of the lunar inequality lead to nearly the same results. Recent observed values have mostly been about $9''{\cdot}207$. Corresponding to Spencer Jones's calculated value, the amplitude in longitude would be $6''{\cdot}872$. A determination by E. P. Fedorov (1952) separates the corrections to the amplitudes in obliquity and longitude, and indicates that the former should be $9''{\cdot}190$ to $9''{\cdot}200$, which would agree with a correcting factor of 0·996 to 0·997. In longitude he gets $6''{\cdot}874$ to $6''{\cdot}886$, larger than the rigid body value. But theoretically both components should be reduced, that in longitude by the larger factor. There is therefore still an inconsistency.

8. *The bodily tide numbers for diurnal tides.*—To get these we have to express ζ_0 in terms of k/ω^2 and then express r_1+ir_2, s_1+is_2 in terms of c_1+ic_2. The results are as follows:

n/ω	h	k	l	$1-h+k$	$1+h-\frac{3}{2}k$
$-\frac{1}{13\cdot7}$	0·590	0·244	0·082	0·654	1·224
$-\frac{1}{183}$	0·523	0·218	0·084	0·695	1·196
$-\frac{1}{6800}$	0·490	0·205	0·086	0·715	1·182
0	0·492	0·206	0·086	0·714	1·183
$\frac{1}{6800}$	0·494	0·207	0·086	0·713	1·184
$\frac{1}{183}$	0·555	0·231	0·082	0·676	1·209
$\frac{1}{13\cdot7}$	0·584	0·242	0·082	0·658	1·221

l is practically the same as on the statical theory. h and k are substantially less, except that h approaches the statical value for the tides O and OO. The statical values of $1-h+k$ and $1+h-\frac{3}{2}k$ are 0·704 and 1·152. These are the factors associated with observations of the bodily tide, and it appears that K_1 and its companions are likely to be nearly the same as on the statical theory, but there is some hope of detecting a difference for O.

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THE THEORY OF NUTATION AND THE VARIATION OF LATITUDE: THE ROCHE MODEL CORE

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Summary

The theory given in a previous paper is modified by the use of a core with the density a quadratic function of the radius, the variation of density being wholly due to pressure. Three pairs of cubic terms are introduced into the displacements. The period of the Eulerian nutation would be 395 days, and the amplitude of the 19-yearly nutation would be multiplied by a factor 0·9975. Allowance for the effect of the ocean would increase the former period to about 430 days, and in both respects there is now tolerable agreement with observa-ations. Correcting factors are given for the other principal nutations. Those for the fortnightly nutation do not differ much from those for the central particle model. There are substantial changes for the semiannual nutations, owing to changes in free periods that occur in their neighbourhoods. The 19-yearly nutation in longitude remains anomalous.

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1. In a previous paper (Jeffreys and Vicente, 1957) we have used Takeuchi's solution for the shell; the core was replaced by homogeneous incompressible fluid with an extra particle at the centre, adjusted so that the mass and moment of inertia of the core agree with those used in Bullard's theory of the figure of the Earth. In the present paper we consider as an alternative a core with a distribution of density of Roche type,

$$\rho_0 = k_0 - k_1 r^2/a_1^2, \tag{1}$$

the variation of density being considered due wholly to compressibility. The displacements are no longer linear functions of the coordinates. In a previous paper (Jeffreys, 1950) a suitable set was introduced, containing cubic functions of the coordinates. This is somewhat modified in the present paper.

The change of density at any point is $-u_i(\partial\rho_0/\partial x_i) - \rho_0(\partial u_i/\partial x_i)$. If the core boundary is not deformed and there is no change of density at internal points, the gravitational potential due to the core is unaltered. In the central particle model the (l_1, m_1) displacement is along level surfaces and satisfies these conditions; but since in the Roche model the ellipticities of the level surfaces are not all equal this might need modification. It is found that independent displacements satisfying

$$u_i \frac{\partial \rho_0}{\partial x_i} = 0\,; \quad \frac{\partial u_i}{\partial x_i} = 0 \tag{2}$$

are, to order k_1,

$$\left.\begin{aligned} v_{1i} &= z\left\{1 + e\left(1 - \frac{6k_1}{35k_0} + \frac{6k_1}{35k_0}\frac{r^2}{a_1^2}\right) + \tfrac{12}{35}\frac{ek_1z^2}{k_0a_1^2}\right\}(l_1, m_1) \quad i = 1, 2 \\ \text{and} \quad v_{1i} &= -(l_1x + m_1y)\left\{1 - e\left(1 - \frac{6k_1}{35k_0} + \frac{6k_1}{35k_0}\frac{r^2}{a_1^2}\right) + \tfrac{12}{35}\frac{ek_1z^2}{k_0a_1^2}\right\} \quad i = 3 \end{aligned}\right\} \tag{3}$$

$$\left.\begin{aligned} v_{3i} &= \{r^2z(1+e) + 2ez^3\}\left(\frac{l_3, m_3}{a_1^2}\right) \quad i = 1, 2. \\ v_{3i} &= -\frac{l_3x + m_3y}{a_1^2}\{r^2(1-e) + 2ez^2)\} \quad i = 3 \end{aligned}\right\} \tag{4}$$

A further simplification may be made, for we are neglecting k_1^2 and e^2 and may therefore naturally neglect $k_1 e$. Then v_{1i} reduces after all to the form for a uniform core. In v_{3i} we have already dropped k_1, since v_{3i} is a consequence of non-uniformity in density and its reaction on l_1, m_1 will contain a factor k_1; we retain e, however, because l_3/l_1, m_3/m_1 are not small in some motions. These two parts generalize the (l_1, m_1) displacements of the homogeneous case.

In addition we must generalize the (l_2, m_2) displacement to include some cubic terms. If the change of density is

$$\Delta\rho = \frac{1}{a_1^2}\left(\rho_1 + \rho_2 \frac{r^2}{a_1^2}\right) K_2, \tag{5}$$

and the normal displacement at the boundary is

$$u_n = \frac{q(\alpha)}{a_1} K_2, \tag{6}$$

there is effectively an extra surface density

$$\sigma = (k_0 - k_1) u_n = (k_0 - k_1) \frac{q(\alpha)}{a_1} K_2. \tag{7}$$

The change of external potential is found to be

$$U_e = 4\pi f \{\tfrac{1}{5}(k_0 - k_1) q(\alpha) + \tfrac{1}{35}\rho_1 + \tfrac{1}{45}\rho_2\} \left(\frac{a_1}{r}\right)^5 K_2, \tag{8}$$

and that of internal potential is

$$U_c = \pi f \left\{\tfrac{4}{5}(k_0 - k_1) q(\alpha) + \left(\tfrac{2}{5} - \tfrac{2}{7}\frac{r^2}{a_1^2}\right)\rho_1 + \left(\tfrac{1}{5} - \tfrac{1}{9}\frac{r^4}{a_1^4}\right)\rho_2\right\} K_2. \tag{9}$$

If the displacements are

$$u_i = \left(\beta + \gamma \frac{r^2}{a_1^2}\right) \frac{\partial K_2}{\partial x_i} + \frac{\delta x_i}{a_1^2} K_2, \tag{10}$$

we find

$$\Delta\rho = -\{k_0(4\gamma + 5\delta) - 4\beta k_1\} \frac{K_2}{a_1^2} + k_1(8\gamma + 7\delta)\frac{r^2}{a_1^4} K_2. \tag{11}$$

$$u_n = \frac{x_i}{r} u_i = \left\{2\beta + (2\gamma + \delta)\frac{r^2}{a_1^2}\right\} \frac{K_2}{r}. \tag{12}$$

For the choice of component displacements we have the following considerations. $q(\alpha)$ is a shell coordinate and may naturally be adopted also as a core coordinate. We can take a linear form to give this. We take two other forms, both giving no normal displacement at the boundary, of which one contributes nothing to $\partial u_i/\partial x_i$, but gives a radial displacement at internal points, while the other contributes to $\partial u_i/\partial x_i$, but gives no radial displacement at internal points. These are all essentially similar to the elastic displacements in the shell and their products by e can be neglected.

We take (writing only one harmonic)

$$v_{2i} = \tfrac{1}{2} q(\alpha) \frac{\partial K_2}{\partial x_i}, \tag{13}$$

$$v_{4i} = l_4 \left\{\left(1 - \tfrac{5}{3}\frac{r^2}{a_1^2}\right)\frac{\partial K_2}{\partial x_i} + \tfrac{4}{3}\frac{x_i}{a_1^2} K_2\right\}, \tag{14}$$

$$v_{5i} = l_5 \left(\frac{r^2}{a_1^2}\frac{\partial K_2}{\partial x_i} - 2\frac{x_i}{a_1^2} K_2\right). \tag{15}$$

Then

$$\frac{\partial u_i}{\partial x_i} = -\frac{6l_5}{a_1^2} K_2, \tag{16}$$

$$qK_2 = x_i u_i = ru_n = \left\{q(\alpha) + 2l_4\left(1 - \frac{r^2}{a_1^2}\right)\right\}K_2, \tag{17}$$

$$\rho_1 = 2k_1 q(\alpha) + 4k_1 l_4 + 6k_0 l_5 = 2k_1 q(\alpha) + \sigma_1, \tag{18}$$

$$\rho_2 = -4k_1 l_4 - 6k_1 l_5, \tag{19}$$

$$\tfrac{1}{35}\rho_1 + \tfrac{1}{45}\rho_2 = \frac{2k_1}{35} q(\alpha) + \tfrac{8}{315} k_1 l_4 + \tfrac{6}{35}(k_0 - \tfrac{7}{9}k_1)l_5. \tag{20}$$

Then, up to cubic terms,

$$u_i' = v_{1i} + v_{2i} + v_{3i} + v_{4i} + v_{5i}. \tag{21}$$

We need also an expression for λ, the bulk-modulus. Since the variation of density is attributed entirely to compression, we have

$$\lambda = \rho\frac{dp}{d\rho} = \rho^2\frac{d\Psi}{d\rho} = \tfrac{2}{3}\pi f a_1^2 \frac{k_0^3}{k_1}\left(1 - \frac{k_1}{k_0}\xi_c^2\right)^2\left(1 - \tfrac{3}{5}\frac{k_1}{k_0}\xi_c^2\right), \tag{22}$$

where $\xi_c = r/a_1$. $(\lambda/\rho)^{1/2}$ varies from 8·7 km/sec at the centre to 6·2 km/sec at the boundary. The seismological values are 11·3 and 8·1 km/sec. The difference is due to the use of a Roche model, which takes no account of the inner core. With this model, subject to the mass and moment of inertia of the core, the above expression for λ cannot be altered. The correct values of the compressibility are about 0·6 of those given by the Roche model, but since we have already the solution for the incompressible model we may expect that interpolation will give a result not far from the truth. In integration we must use the full expression for λ, since, though k_1/k_0 is treated as small, $2{\cdot}6\,k_1/k_0$ cannot be.

2. *The potential terms.*—We write as before $c = 4\pi f D$, and find

$$\begin{aligned}\iint \rho_0 u_n'(2U_1 - g_1 u_n' + U_c)dS \\ = 4\pi f(k_0 - k_1)Ma_1^5 q(\alpha)\{2D - \tfrac{2}{15}(k_0 - \tfrac{3}{7}k_1)q(\alpha) + \tfrac{8}{315}k_1 l_4 \\ + \tfrac{6}{35}(k_0 - \tfrac{7}{9}k_1)l_5\},\end{aligned} \tag{1}$$

$$\begin{aligned}-\iiint\left(2U_1 + u_k'\frac{\partial\Psi}{\partial x_k} + U_c\right)\left(\rho\frac{\partial u_i'}{\partial x_i} + u_i'\frac{\partial\rho_0}{\partial x_i}\right)d\tau \\ = -4\pi f Ma_1^5[2D\{\tfrac{2k_1}{7}q(\alpha) + \tfrac{8}{63}k_1 l_4 + \tfrac{6}{7}(k_0 - \tfrac{7}{9}k_1)l_5\} \\ + \tfrac{4}{105}k_0 k_1 q^2(\alpha) - \tfrac{8}{315}k_0 k_1 q(\alpha) l_4 \\ + \tfrac{4}{35}k_0(k_0 - \tfrac{4}{3}k_1)q(\alpha)l_5 + \tfrac{16}{297}k_0 k_1 l_4^2 - \tfrac{8}{63}k_0^2 l_4 l_5 + \tfrac{8}{35}k_0^2 l_2^5],\end{aligned} \tag{2}$$

$$\begin{aligned}-\iiint \rho_0 u_i'\frac{\partial\Psi}{\partial x_i}\frac{\partial u_k'}{\partial x_k}d\tau = -8\pi f Ma_1^5 l_5[(\tfrac{1}{7}k_0^2 - \tfrac{8}{45}k_0 k_1)q(\alpha) \\ + 2l_4(\tfrac{2}{63}k_0^2 - \tfrac{16}{495}k_0 k_1)],\end{aligned} \tag{3}$$

$$-\iiint\lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2 d\tau = -4\pi f Ma_1^5 \,.\, 265 l_5^2, \tag{4}$$

where k_0 and k_1 have been given their numerical (c.g.s.) values in the last expression.

We have also

$$DP = 2{\cdot}056q(\alpha) + 1{\cdot}723l_5 + 0{\cdot}0935l_4, \tag{5}$$

$$DK'(\alpha+) = -9{\cdot}179DP + 10{\cdot}152q(\alpha). \tag{6}$$

The shell terms given in 4 (36) (37) (38) of the previous paper must be expressed in terms of $q(1)$, $q(\alpha)$, l_4 and l_5 and added to these core terms. The total is

$$\begin{aligned}2L = -Ma^3 \times 10^{12}[&8{\cdot}415q^2(1) - 6{\cdot}353q(1)q(\alpha) - 1{\cdot}5369Dq(1)\\ &- 0{\cdot}0233q(1)l_5 - 0{\cdot}00126q(1)l_4 - 0{\cdot}5740Dq(\alpha)\\ &+ 2{\cdot}138q^2(\alpha) + 0{\cdot}0373q(\alpha)l_4 + 0{\cdot}629q(\alpha)l_5\\ &- 0{\cdot}0167Dl_4 + 0{\cdot}0239l_4^2\\ &- 0{\cdot}389Dl_5 - 0{\cdot}2368l_4l_5 + 4{\cdot}58l_5^2 - 0{\cdot}0744D^2].\end{aligned} \tag{7}$$

This is stationary when

$$q(1) = 0{\cdot}324D; \quad q(\alpha) = 0{\cdot}618D; \quad l_4 = -0{\cdot}129D; \quad l_5 = -0{\cdot}0025D.$$

These lead to the bodily tide numbers, for the statical solution,

$$h = 0{\cdot}598; \quad k = 0{\cdot}273; \quad l = 0{\cdot}082. \tag{8}$$

In comparison with the central particle model, $q(1)$ and $q(\alpha)$ are slightly increased, but l_4 and l_5 make small negative contributions to k, which is slightly diminished. l is unaltered.

However, it is easy to see that the differences can represent nothing but minor errors of approximation. We are considering similar constitutions of the shell; and the core in the two models has the same mass and moment of inertia. Suppose the same displacements applied to the shell in both models. Then the potential acting on the core from outside it is the same for both, and therefore, by the Radau approximation, the level surface of mean radius a_1 is the same in both. But this implies that U_e outside the core is the same and therefore the equations of equilibrium of the shell hold; and so do the stress conditions.

By using the condition that Ψ_0 is a function of ρ_0 we can prove that the sum of (3) and (4) is exactly

$$-\iiint\left(\rho_0\frac{\partial u_i'}{\partial x_i} + u_i'\frac{\partial\rho_0}{\partial x_i}\right)^2 \frac{d\Psi_0}{d\rho_0}\,d\tau; \tag{9}$$

and in (2) the terms $U_e + u_k'\partial\Psi/\partial x_k$ give the change of potential at a given particle in its new position. Hence where an equilibrium theory is applicable the usual theory of the figure of the Earth can be adapted to the present treatment. Examination of this theory showed that the terms in the ellipticity that contain k_1^2/k_0^2 have coefficients of the order of $\frac{1}{10}$. Hence neglect of these terms may lead to less inaccuracy than might have been expected.

As an additional check the statical case has been treated by solution of Clairaut's differential equation for a Roche model core. The elevation of a surface of constant density is taken as $\epsilon K_2/a$, and if the central value of ϵ is ϵ_a the result is

$$\epsilon_a = 0{\cdot}264D, \quad q(\alpha) = 0{\cdot}604D, \quad q(1) = 0{\cdot}319D.$$

The results agree with (8) as closely as need be expected.

Thus the whole of the equilibrium conditions are satisfied to the first order in the departures from homogeneity; and any difference must be due to the neglect of square terms. Since corresponding terms have been retained in the treatment of the central particle model the solution for this is the more accurate.

3. *Terms containing the velocities.*—We first define the following core integrals.

$$\iiint \rho_0(x^2+z^2)d\tau = A_1, \quad \iiint \rho_0(x^2+y^2)d\tau = C_1, \quad \iiint \rho_0 \frac{r^2(x^2+z^2)}{a_1^2}\, d\tau = A_{11},$$

$$\iiint \rho_0 \frac{r^4(x^2+z^2)}{a_1^4} = A_{12}; \text{ etc.}; \tag{1}$$

$$\bar{A}_1 = \iiint \rho_0 \tfrac{2}{3} r^2 d\tau = \tfrac{8}{15}\pi a_1^5(k_0 - \tfrac{7}{5}k_1), \tag{2}$$

$$A_{11} = \iiint \rho_0 \tfrac{2}{3} \frac{r^4}{a_1^2}\, d\tau = \tfrac{8}{21}\pi a_1^5(k_0 - \tfrac{7}{9}k_1), \tag{3}$$

$$A_{12} = \iiint \rho_0 \tfrac{2}{3} \frac{r^6}{a_1^4}\, d\tau = \tfrac{8}{27}\pi a_1^5(k_0 - \tfrac{9}{11}k_1), \tag{4}$$

$$C_1 - A_1 = \tfrac{8}{15}\pi \int \rho_0 d(r^5 e') = \tfrac{8}{15}\pi e a_1^5 \int (k_0 - k_1\xi_c^2)\left\{d\xi_c^5 - \tfrac{6}{35}\frac{k_1}{k_0}\, d\xi_c^5(1-\xi_c^2)\right\} \tag{5}$$

In the second part of (5), the integral of the product by k_0 is zero, and that of the product by $k_1\xi_c^2$ is of order k_1^2. Hence

$$C_1 - A_1 = eA_1 + A_1 O(k_1^2 e), \tag{6}$$

and similarly to the same accuracy

$$C_{11} - A_{11} = eA_{11}, \quad C_{12} - A_{12} = eA_{12}. \tag{7}$$

Also, if we neglect e,

$$\iiint \rho_0 \frac{z^2 r^2}{a_1^2}\, d\tau = \tfrac{1}{2}A_{11}, \tag{8}$$

$$\iiint \rho_0 \frac{z^4}{a_1^2}\, d\tau = \tfrac{3}{10}A_{11}, \quad \iiint \rho_0 \frac{z^2 x^2}{a_1^2}\, d\tau = \tfrac{1}{10}A_{11}, \tag{9}$$

$$\iiint \rho_0 \frac{z^2 r^4}{a_1^4}\, d\tau = \tfrac{1}{2}A_{12}, \quad \iiint \rho_0 \frac{r^2 z^4}{a_1^4}\, d\tau = \tfrac{3}{10}A_{12}, \quad \iiint \rho_0 \frac{r^2 z^2 x^2}{a_1^4}\, d\tau = \tfrac{1}{10}A_{12}. \tag{10}$$

We find

$$\iiint \rho_0 \dot{u}_i'^2 d\tau = A_1(\dot{l}_1^2 + \dot{m}_1^2) + 2(1+\tfrac{4}{5}e)A_{11}(\dot{l}_1\dot{l}_3 + \dot{m}_1\dot{m}_3) + (1+\tfrac{8}{5}e)A_{12}(\dot{l}_3^2 + \dot{m}_3^2), \tag{11}$$

$$\iiint 2\rho_0 \dot{u}_i' \dot{u}_{\omega i}\, d\tau = 2A_1(\dot{l}\ddot{l}_1 + \dot{m}\dot{m}_1) + 2(1+\tfrac{4}{5}e)A_{11}(\dot{l}\ddot{l}_3 + \dot{m}\dot{m}_3), \tag{12}$$

$$\iiint 2\rho_0\omega\{z(\dot{m}u_1' - \dot{l}u_2') - u_3'(\dot{m}x - \dot{l}y)\}d\tau$$

$$= 2\omega A_1(\dot{m}l_1 - \dot{l}m_1) + 2\omega A_{11}(1+\tfrac{4}{5}e)(\dot{m}l_3 - \dot{l}m_3), \tag{13}$$

$$\iiint 2\rho_0\omega(u_1'\dot{u}_2' - u_2'\dot{u}_1')d\tau \equiv \omega(1+e)[A_1(l_1\dot{m}_1 - m_1\dot{l}_1) + A_{11}\left(1 + \frac{6e}{5}\right)$$

$$\times (l_1\dot{m}_3 + l_3\dot{m}_1 - m_1\dot{l}_3 - m_3\dot{l}_1) + A_{12}(1+\tfrac{12}{5}e)(l_3\dot{m}_3 - m_3\dot{l}_3)] + \omega A_1(\dot{m}_1 s_1 - \dot{l}_1 s_2)$$

$$+ \omega A_{11}(\dot{m}_3 s_1 - \dot{l}_3 s_2) + 2\omega(A_1 - \tfrac{7}{5}A_{11})(\dot{m}_1 l_4 - \dot{l}_1 m_4) + 2\omega(A_{11} - \tfrac{7}{5}A_{12})(\dot{m}_3 l_4 - \dot{l}_3 m_4)$$

$$+ \tfrac{6}{5}\omega A_{11}(\dot{m}_1 l_5 - \dot{l}_1 m_5) + \tfrac{6}{5}\omega A_{12}(\dot{m}_3 l_5 - \dot{l}_3 m_5), \tag{14}$$

where the usual approximations have been made; the sign $\equiv$ indicates that a derivative with regard to t has been dropped.

There is evidently an advantage in adopting a new pair of variables

$$(l_6, m_6) = (l_1, m_1) + (1 + \tfrac{4}{5}e)\frac{A_{11}}{A_1}(l_3, m_3). \tag{15}$$

Then the terms in $2L$ that contain the velocities or $C-A$ are

$$\begin{aligned}
&\equiv A(\dot{l}^2+\dot{m}^2)+\omega(2A-C)(l\dot{m}-\dot{l}m)-\omega^2(C-A)(l^2+m^2)-2(C-A)(lc_1+mc_2)\\
&\quad+A_1(\dot{l}_6^2+\dot{m}_6^2)+2A_1(\dot{l}\dot{l}_6+\dot{m}\dot{m}_6)+2\omega A_1(\dot{m}l_6-\dot{l}m_6)+(1+\tfrac{8}{5}e)(A_{12}-A_{11}^2/A_1)(\dot{l}_3^2+\dot{m}_3^2)\\
&\quad+A_1\omega(1+e)(l_6\dot{m}_6-m_6\dot{l}_6)+\tfrac{4}{5}\omega eA_{11}(l_6\dot{m}_3-m_6\dot{l}_3)+\omega(A_{12}-A_{11}^2/A_{12})(1+\tfrac{17}{5}e)\\
&\qquad\qquad\times(l_3\dot{m}_3-m_3\dot{l}_3)\\
&\quad+\omega A_1(\dot{m}_6s_1-\dot{l}_6s_2)+2\omega(A_1-\tfrac{7}{5}A_{11})(\dot{m}_6l_4-\dot{l}_6m_4)-\tfrac{14}{5}\omega(A_{12}-A_{11}^2/A_1)(\dot{m}_3l_4-\dot{l}_3m_4)\\
&\quad+\tfrac{6}{5}\omega A_{11}(\dot{m}_6l_5-\dot{l}_6m_5)+\tfrac{6}{5}\omega(A_{12}-A_{11}^2/A_1)(\dot{m}_3l_5-\dot{l}_5m_5).
\end{aligned}\tag{16}$$

Then in all

$$\begin{aligned}
\frac{2L}{C}&=(1-H)(\dot{l}^2+\dot{m}^2)+(1-2H)\omega(l\dot{m}-\dot{l}m)-\omega^2H(l^2+m^2)-2H(c_1l+c_2m)\\
&\quad+0{\cdot}107[\dot{l}_6^2+\dot{m}_6^2+2(\dot{l}\dot{l}_6+\dot{m}\dot{m}_6)+0{\cdot}04763\omega(1+\tfrac{8}{5}e)(\dot{l}_3^2+\dot{m}_3^2)+2\omega(\dot{m}l_6-\dot{l}m_6)\\
&\qquad+(1+e)\omega(\dot{m}_6l_6-m_6\dot{l}_6)+0{\cdot}55844e\omega(l_6\dot{m}_3-m_6\dot{l}_3)\\
&\qquad+0{\cdot}04763\omega(1+\tfrac{17}{5}e)(l_3\dot{m}_3-m_3\dot{l}_3)+\omega(\dot{m}_6s_1-\dot{l}_6s_2)\\
&\qquad+0{\cdot}04546\omega(\dot{m}_6l_4-\dot{l}_6m_4)-0{\cdot}1334\omega(\dot{m}_3l_4-\dot{l}_3m_4)\\
&\qquad+0{\cdot}83766\omega(\dot{m}_6l_5-\dot{l}_6m_5)+0{\cdot}05716\omega(\dot{m}_3l_5-\dot{l}_3m_5)\\
&\quad+\omega^2[-425{\cdot}9(r_1^2+r_1^2)+321{\cdot}5(r_1s_1+r_2s_2)+77{\cdot}78(d_1'r_1+d_2'r_2)\\
&\qquad+1{\cdot}179(r_1l_5+r_2m_5)+0{\cdot}0638(r_1l_4+r_2m_4)+29{\cdot}05(d_1's_1+d_2's_2)\\
&\qquad-108{\cdot}2(s_1^2+s_2^2)-1{\cdot}888(s_1l_4+s_2m_4)-31{\cdot}83(s_1l_5+s_2m_5)\\
&\qquad-1{\cdot}210(l_4^2+m_4^2)+11{\cdot}98(l_4l_5+m_4m_5)-231{\cdot}8(l_5^2+m_5^2)\\
&\qquad+0{\cdot}845(d_1'l_4+d_2'l_5)+19{\cdot}69(d_1'l_5+d_2'm_5)+3{\cdot}765(d_1'^2+d_2'^2)]
\end{aligned}\tag{17}$$

$r_1, r_2, s_1, s_2, l_4, m_4, l_5, m_5$ are quasi-statical coordinates.

We find in turn

$$(r_1, r_2)=0{\cdot}3776(s_1, s_2)+0{\cdot}09131(d_1', d_2')+0{\cdot}0014(l_5, m_5)+0{\cdot}0000075(l_4, m_4), \tag{18}$$

$$\begin{aligned}(s_1, s_2)&=0{\cdot}6147(d_1', d_2')-0{\cdot}3303(l_5, m_5)-0{\cdot}0196(l_4, m_4)\\ &\qquad+0{\cdot}001126(\dot{m}_6/\omega, -\dot{l}_6/\omega),\end{aligned}\tag{19}$$

$$\begin{aligned}(l_5, m_5)&=0{\cdot}0001196(\dot{m}_6/\omega, -\dot{l}_6/\omega)+0{\cdot}00001349(\dot{m}_3/\omega, -\dot{l}_3/\omega)\\ &\qquad+0{\cdot}00093(d_1', d_2')+0{\cdot}0278(l_4, m_4),\end{aligned}\tag{20}$$

$$(l_4 . m_4)=0{\cdot}00210(\dot{m}_6/\omega, -\dot{l}_6/\omega)-0{\cdot}00693(\dot{m}_3/\omega, -\dot{l}_3/\omega)-0{\cdot}139(d_1', d_2'). \tag{21}$$

Forming the modified Lagrangian and eliminating d_1', d_2' by

$$(d_1', d_2')=0{\cdot}006345\left(\frac{(c_1, c_2)}{\omega^2}+\frac{(\dot{m}, -\dot{l})}{\omega}+(l, m)\right) \tag{22}$$

we have

$$\begin{aligned}\frac{2L'}{C} &= (1 - H + 0{\cdot}001018)(\dot l^2 + \dot m^2) + (1 - 2H + 0{\cdot}002036)\omega(l\dot m - \dot l m)\\
&\quad - (H - 0{\cdot}001018)\omega^2(l^2 + m^2) - 2(H - 0{\cdot}001018)(c_1 l + c_2 m)\\
&\quad + \frac{0{\cdot}002036}{\omega}(c_1\dot m - c_2\dot l) + 0{\cdot}1070679(\dot l_6^2 + \dot m_6^2) - 0{\cdot}0000289(\dot l_6\dot l_3 + \dot m_6\dot m_3)\\
&\quad + \{0{\cdot}005096(1 + \tfrac{8}{5}e) + 0{\cdot}0000489\}(\dot l_3^2 + \dot m_3^2)\\
&\quad + 0{\cdot}21443(\dot l\dot l_6 + \dot m\dot m_6) + 0{\cdot}21443(l_6\dot m - m_6\dot l) + 0{\cdot}107(1 + e)\omega(l_6\dot m_6 - m_6\dot l_6,\\
&\quad + 0{\cdot}05975e\omega(l_6\dot m_3 - m_6\dot l_3) + 0{\cdot}005096\omega(1 + \tfrac{17}{5}e)(l_3\dot m_3 - m_3\dot l_3)\\
&\quad + \frac{0{\cdot}0004251}{\omega}(c_1\dot m_6 - c_2\dot l_6) + 0{\cdot}00001245(\dot l\dot l_3 + \dot m\dot m_3)\\
&\quad + \frac{0{\cdot}00001245}{\omega}(c_1\dot m_3 - c_2\dot l_3) + 0{\cdot}00001245\omega(l\dot m_3 - m\dot l_3).\end{aligned} \tag{23}$$

We put

$$l + im = \zeta e^{i\gamma t}, \quad l_6 + im_6 = \zeta_6 e^{i\gamma t}, \quad l_3 + im_3 = \zeta_3 e^{i\gamma t}. \tag{24}$$

The equations of motion lead to three free speeds 0, $-\omega$, $-\omega$, as was to be expected. The other three are near these values, namely

$$\gamma = 0{\cdot}002531\omega, \quad \zeta_6 = -0{\cdot}99940\zeta, \quad \zeta_3 = +0{\cdot}014\zeta \tag{25}$$

$$\gamma = -\omega + n; \quad n = -0{\cdot}00403\omega, \quad \zeta_6 = -9{\cdot}326\zeta, \quad \zeta_3 = -18{\cdot}4\zeta \tag{26}$$

$$\gamma = -\omega + n; \quad n = +0{\cdot}00679\omega, \quad \zeta_6 = -9{\cdot}333\zeta, \quad \zeta_3 = +87{\cdot}8\zeta. \tag{27}$$

The first makes the period of the Eulerian nutation 394·9 days, slightly longer than for the central particle model. The slip at the boundary is essentially $\zeta_1 + \zeta_3 = \zeta_6 + \frac{2}{7}\zeta_3 = -0{\cdot}9990\zeta$ (with a small correction due to ellipticity of the boundary).

The second corresponds to the second free period for the central particle model. n is nearly doubled. The third is a consequence of the variation of density in the core and is a completely new feature. ζ_6 corresponds to ζ_1 of the central particle model and has nearly the same value. The striking result is that the slip in the second is much increased, and in the third is reversed. The cubic terms in the displacements are therefore of great importance. The term $0{\cdot}0000489\ (\dot l_3^2 + \dot m_3^2)$ in the modified Lagrangian has arisen mainly in the elimination of l_4, m_4, and the results are very sensitive to the difference between this coefficient and that of $\omega(l_3\dot m_3 - m_3\dot l_3)$, which otherwise is a multiple of e. Physically the explanation is that l_4, m_4 includes a radial displacement, and consequently would imply a change of angular momentum, unless it was compensated by a change of transverse velocity. This can be verified by considering the behaviour of (17) when $k_1 \to 0$. All the l_4, m_4 terms not containing the velocities tend to zero. The matter tends to incompressibility and the coefficient of $l_5^2 + m_5^2$ tends to infinity. Thus $l_5, m_5 \to 0$, and then the equations of motion for l_4, m_4 give $\dot l_3 = 0$, $\dot m_3 = 0$. Thus l_3, m_3 are essentially a consequence of non-uniformity of density.

The solutions for the principal forced nutations are in Table I, ζ_0 being the value for a rigid Earth in each case. The values are means of our solutions and a set made at the Mathematical Laboratory, Cambridge; the differences were within the possible effects of rounding-off errors.

TABLE I

n/ω	ζ/ζ_0	ζ_6/ζ_0	ζ_3/ζ_0	Slip/ζ_0
$-1/13{\cdot}7$	1·067	−0·415	+0·274	−0·29
$-1/183$	1·142	−1·312	−2·03	−1·89
$-1/6800$	0·99889	0·0104	0·047	0·023
0	1	0	0	0
$1/6800$	1·00120	−0·0112	−0·042	−0·023
$1/183$	0·9707	0·258	−4·61	−1·0
$1/13{\cdot}7$	1·0266	−0·460	0·482	−0·32

The slip is $\zeta_6+(1-A_{11}/A_1)\zeta_3$ or, nearly, $\zeta_6+\frac{2}{7}\zeta_3$.

We write $l_4+im_4=\zeta_4$, $l_5+im_5=\zeta_5$; the solution for the remaining variables is in Table II.

TABLE II

n/ω	$q(1)/D$	$q(\alpha)/D$	ζ_4/D	ζ_5/D	h	k	l	$1-h+k$	$1+h-\frac{3}{2}k$
$-\frac{1}{13{\cdot}7}$	0·324	0·617	−0·126	−0·002	0·597	0·258	0·070	0·661	1·210
$-\frac{1}{183}$	0·385	0·779	−1·433	0·002	0·710	0·298	0·072	0·588	1·263
0	0·299	0·552	+0·799	0·001	0·551	0·244	0·082	0·693	1·185
$\frac{1}{183}$	0·308	0·574	+2·922	0·027	0·568	0·264	0·084	0·696	1·172
$\frac{1}{13{\cdot}7}$	0·327	0·625	−0·169	−0·003	0·603	0·261	0·078	0·658	1·211

The values of $1-h+k$ and $1+h-\frac{3}{2}k$ for semidiurnal and long-period tides are 0·675 and 1·188. On the whole the behaviour is very similar to that for the central particle model. The chief differences are an appreciable increase of yielding for all the K_1 tides and a considerable change in some of the variables for the P tides. The latter are connected with the change of the free period near $n/\omega=-1/183$ and the introduction of a new free period near $n/\omega=+1/183$. The theoretical amplitude of the semiannual nutation in obliquity is 0″·55; if this could be measured it would give useful information.

4. The Roche model used certainly overestimates the compressibility, and better theoretical values will usually be found by interpolating; if x denotes an estimate for the central particle model and y that for the Roche model, $0{\cdot}4x+0{\cdot}6y$ should be better than either. This will not apply, however, to the semiannual nutations and P tides, since the effects of changes of the free periods cannot be interpolated in any such simple way.

The correction to the 19-yearly nutation is reduced, and the correcting factor should be close to $-0{\cdot}0022$. The coefficient would be about 9″·207, in fair agreement with most observed values, though the difficulty about the ratio of the amplitudes in obliquity and longitude persists.

Both models are imperfect in so far as they do not allow for the size of the inner core. Allowance for this would introduce much greater difficulties. For a homogeneous incompressible core, and with some accuracy for a core deviating somewhat from this condition, the solution depends on the differential equation

$$\gamma^2\left(\frac{\partial^2\Omega}{\partial x^2}+\frac{\partial^2\Omega}{\partial y^2}\right)+(\gamma^2-4\omega^2)\frac{\partial^2\Omega}{\partial z^2}=0. \tag{28}$$

For $\omega=0$ solutions can be built up from $1/r$ and its derivatives, and have no singularity except at $r=0$. But when $|\gamma|<2\omega$ this procedure leads to functions

with infinities on the cones $z = \pm(4\omega^2/\gamma^2 - 1)^{1/2}(x^2+y^2)^{1/2}$, and these would extend right through the fluid. The discovery of non-singular solutions between two spheres would resemble the treatment of oscillations in an elliptic lake; they will be of exponential type over some ranges of latitude and oscillatory over others. Whereas the effects of an inner sphere, in potential problems, are of the order of the fifth power of the ratio of the radii, they may here be of the second or third. Since the radius of the inner core is 0·36 of that of the main core the effect is probably only of the order of a tenth of that of the core as a whole.

5. *The effect of the ocean.*—Larmor (1915) and one of us (1915) noticed at almost the same time that the effect of the ocean on the period of the Eulerian nutation is not negligible.* It could be allowed for in the present problem as follows. The period being long, a statical treatment is assumed permissible. The elevation of the ocean surface can be introduced as an additional coordinate, contributing to the potential energy, and a new numerical solution can be made. However, this hardly seems necessary. The effect of the fluidity of the core by itself would be to reduce the period by about 30 days, and the increase due to the elasticity of the shell is about 120 days. Thus elasticity is much the more important. But for an entirely elastic body the effect on the period depends wholly on Love's number k, and we can use any model that reproduces this correctly. We therefore consider a homogeneous incompressible Earth of rigidity k, covered by a shallow ocean. The theory for the solid is well known; for the ocean we add the conditions that the free surface is an equipotential and that the normal stress is continuous at the interface. We denote the normal displacements at the interface and the ocean surface by $q(1)K_2/a$, $q(1+)K_2/a$, the densities by ρ and $\rho(1+\eta)$, and write

$$U_1' = k_2K_2; \quad \frac{\mu}{g\rho(1+\eta)a} = \sigma.$$

We are led to the equations

$$\left(\frac{19}{5}\sigma + \frac{2\eta}{5(1+\eta)}\right)q(1) + \tfrac{2}{5}\,\frac{q(1+)}{1+\eta} = \frac{ak_2}{g},$$

$$\frac{3}{5}\frac{\eta}{1+\eta}q(1) - \frac{\frac{2}{5}+\eta}{1+\eta}q(1+) = -\frac{ak_2}{g},$$

and

$$k = \frac{3}{5}\frac{g}{ak_2}\frac{q(1+)+\eta g(1)}{1+\eta}.$$

Here $1+\eta = 5{\cdot}53$. For the rigid Earth, $\sigma = \infty$, $q(1) = 0$, and the solution is

$$q(1+) = 1{\cdot}1215\, ak_2/g,$$

whence $k = 0{\cdot}1215$ and the free period given by the Larmor–Love formula is 350 days. Thus in this case the ocean lengthens the period by about 45 days.

We have found from our theory, with no ocean, $k = 0{\cdot}289$. A uniform solid Earth would give $k = 0{\cdot}290$ if $\sigma = 0{\cdot}439$, and the free period would be 441 days. But it is better to proceed as follows. The theory, with no ocean, leads to a free period of 392 days. This would correspond, in an entirely elastic body, to $k = 0{\cdot}208$, and the corresponding rigidity would make $\sigma = 0{\cdot}652$. We insert this in the above equations and get

$$q(1) = 0{\cdot}3266, \quad q(1+) = 1{\cdot}3009; \quad k = 0{\cdot}301$$

and the free period is 449 days. Thus the lengthening due to the ocean is 57 days.

* The account of Jeffreys aimed at an estimate of the damping due to viscosity in the ocean; this has of course long been superseded. The lengthening of the free period came out as a by-product. Larmor used an equilibrium theory, which cannot be far wrong for a 14-monthly tide.

Larmor made various rough corrections, allowing for the fact that the ocean does not cover the whole surface and for elasticity. The latter has already been allowed for here. The oceanic effect is likely to be about $\frac{2}{3}$ of that for a complete ocean, namely about 38 days, and the free period will be about 430 days. The uncertainty is probably about ± 5 days. This is sufficiently close to the observed value.

The forced nutations correspond to diurnal tides. There is no satisfactory theory for these in the actual ocean. For an ocean covering the Earth, and for exactly diurnal tides (speed ω) there is no change of level at all, but there are tidal currents. These may contribute to the angular momenta. We may note, however, that the effect on the nutations is likely to be of the order of the ratio of the moments of inertia of the ocean and the body of the Earth, say 1/1000. When the precession is used to compute $(C-A)/C$ an error of this order may be introduced. But for the nutations this factor is likely to be the same within a factor of order n/ω. Hence if the computed value of $(C-A)/C$ is used the correcting factor due to the oceans for the nutations will be of order $\frac{1}{1000}\, n/\omega$ and utterly negligible.

Attention needs to be called to the fact that ζ_0 does not correspond to the nutations given in the standard accounts. Most of these proceed by writing down the equations of motion and neglecting second derivatives with regard to the time. In our definition of ζ_0 these derivatives have been taken into account; thus the standard treatments do not quite give the theoretical motion even for a rigid Earth.

Oppolzer (1886), followed by Woolard, defines the nutation as the motion of the instantaneous axis of rotation. This motion is almost the same as is given by the approximation, but then the motion of the axis of figure relative to that of rotation has to be added as a correction when we are considering displacements of stars relative to the Earth. Woolard (1953) gives the coefficients of the fortnightly terms in obliquity and longitude, for the axis of rotation, as $0''\cdot0884$ and $-0''\cdot2037$. The corresponding values for the axis of figure are $0''\cdot0948$ and $-0''\cdot2211$.

It is worth while to have some indication of how much of the difference from the rigid body theory of the nutations is due to elasticity of the shell and how much to fluidity of the core. This can be obtained from (24) of our previous paper by omitting slip at the core boundary, thus taking $\zeta_1 = 0$. The free period becomes 440 days. For the nutations we should have

$$\zeta = (1 - 0\cdot3045\, n/\omega)\zeta_0.$$

The effect of elasticity is therefore negligible for the 19-yearly nutation, as Poincaré noticed. The extreme value for the periods considered here are:

n/ω	ζ/ζ_0
$-\dfrac{1}{13\cdot7}$	1·022
$\dfrac{1}{13\cdot7}$	0·978

For both components the effect of the core is the remainder, about 0·05 in each case.

6. *The calculated value of the* 19-*yearly nutation.*—Spencer Jones's data on the solar parallax and the lunar inequality led to (Jeffreys, 1948)

$$E/M = 81{\cdot}278 \pm 0{\cdot}025, \quad H = 0{\cdot}00327360 \pm 0{\cdot}00000069.$$

For a rigid Earth these give for the constant of nutation

$$N = 9''{\cdot}2271,$$

in good agreement with Spencer Jones's $9''{\cdot}2272$. (It should be noticed, however, that this depends on using a steady-motion theory; allowance for the nH term in 7(23) of our previous paper would decrease this by about $0''{\cdot}0010$, and there would be a proportionally larger decrease in the longitude term. We shall however neglect this, since the effects that concern us are in the second decimal.) Rabe's parallax (1950) and the lunar inequality from Spencer Jones's data give

$$E/M = 81{\cdot}356; \quad H = 0{\cdot}003327468; \quad N = 9''{\cdot}2242.$$

The change of the solar parallax produces more effect in H than in N.

Applying the corrections $\mp 0{\cdot}0022$ to ζ/ζ_0 for $n/\omega = \pm 1/6800$ we get for the nutation components in obliquity and longitude:

	Spencer Jones's parallax	Rabe's parallax
Obliquity	$9''{\cdot}2120$	$9''{\cdot}2090$
Longitude	$17''{\cdot}210$	$17''{\cdot}204$

Multiplying the longitude terms by $\sin \epsilon$ to give actual displacements gives $6''{\cdot}850$ and $6''{\cdot}848$. The obliquity values by themselves would be in agreement with nearly all observational estimates; but Fedorov's discussion, which is so far the only one that separates the corrections in obliquity and longitude, gives the values $6''{\cdot}886$, $6''{\cdot}874$, $6''{\cdot}874$ for longitude term from three series of observations. Thus the longitude term remains in disagreement.

7. Takeuchi's model for the shell is close to Bullen's Model B; and it may be asked whether permissible changes in the shell model would have appreciable effects, in particular because if the temperature distribution is not adiabatic there will be corresponding changes in the distribution of density and proportional changes in the elastic moduli. This appears to be answered by a recent paper of Bullen (1956), who shows that for his Model A the greatest possible correction to the density at any point is only 0·07 gm/cm^3. The effects of elasticity of the shell are therefore reliable to about 1 per cent.

8. It has been suggested that electromagnetic interaction between the core and the shell may have an appreciable effect. This is not considered here, since the object of this work is to test whether the mechanical properties inferred on other evidence suffice to explain the 14-monthly period and the nutations quantitatively, and thus to provide a comprehensive check on the theory of the mechanical properties. If a definite discrepancy had been found it would have to be considered as a starting-point for further investigations; but as the agreement seems to be within the uncertainties the presumption is that magnetic effects are small.

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THE VARIATION OF DECIMETRE-WAVE RADIATION WITH SOLAR ACTIVITY

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Summary

A statistical method is used to segregate the quiet component from the slowly varying component of solar decimetre-wave radiation in the period 1947–54. For this purpose the radiations at frequencies 2800, 1200, and 600 Mc/s have been correlated with sunspot numbers, sunspot areas, and faculae. The mean lives of the various radiations and activities have been estimated and compared. There is an increase of life with decreasing radio frequency. The life of 2800 Mc/s radiation is about the same as sunspots but measurements of the latter show some anomalies. The slowly varying radiation flux is proportional to frequency in the range studied. The quiet sun flux has a small but significant variation with solar activity, the relative change being greater for smaller frequencies. The possibility that this variation may be associated with uncorrelated local sources, such as prominences, is not entirely excluded.

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1. *Introduction.*—The radiation from the Sun in certain wave-length bands is clearly composed of (i) a quiet component, and (ii) a slowly varying component (1). This is the case, for example, with decimetre radio waves and with the ultra-violet radiations that produce the ionosphere. The regular observations give a measure of the total radiation only and thus call for a method of segregation in order that the two components may be studied individually. In the present paper an attempt is made to segregate the components of decimetre radiation during the period 1947–1954.

The quiet component comes steadily from the whole solar surface and the main question concerning it is whether and how it varies during the sunspot cycle. It is thermal radiation and any variations would represent interesting changes in the normal solar atmosphere. The slowly varying component comes from the local active areas which can be detected by the various visual activity phenomena of sunspots, faculae, etc. We can recognize and measure this component by its correlation with the activity. We are interested in its intensity in relation to the various measures of solar activity, and also in the question of whether the activity and the radiation appear over the same period of time. We are not concerned with the more rapidly varying features—noise storms, bursts and outbursts.

It would be possible in principle to detect and measure all the local emission sources by the use of eclipses or high resolution interferometry. This is difficult to do comprehensively at sunspot maximum, but it is hoped that in due course the conclusions of this paper will be checked by adequate observations of this type.

Our method of segregation was first used by Pawsey and Yabsley (**2**). Later it was used by Christiansen and Hindman (**3**) to demonstrate sunspot cycle changes in decimetre radiation. The method can be explained by reference to Fig. 1, where values of decimetre radiation flux are plotted against simultaneous measures of solar activity. Such plots can be made for any phase of the sunspot (sp-) cycle and extrapolating the resulting curve to zero activity can give the

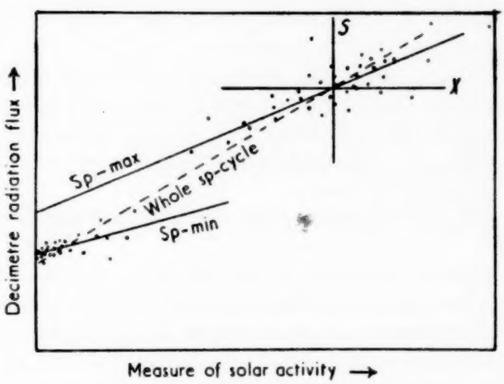

Fig. 1.

quiet component. However we must allow for the following complications of which (*g*) has been regarded as the most important (**4, 5**).

(*a*) There are two regressions representing the linear solution for the points and the true solution may be anywhere between them.
(*b*) The curves may not be linear.
(*c*) There may be changes in the measurement standards during the sp-cycle.
(*d*) The scatter of points may be due in part to the progression of the sp-cycle. This would make the slope steeper as can be seen by the broken line in Fig. 1 representing the slope for the whole sp-cycle.
(*e*) The slope might be affected by measurement changes associated with directivity and the position on the Sun's disk.
(*f*) Some local emissions might be entirely uncorrelated with measurable solar activity.
(*g*) The local emissions and the corresponding solar activity may not occur throughout the same period of time.

These complications were taken into consideration when selecting the data and the form of the correlations.

2. *Data and procedure.*—The radio data used were:

28 The daily flux at 2800 Mc/s as published by Covington (**6, 7**) and multiplied up to the end of 1953 by 1·18 in accordance with (**7,** No. 104).

12 The daily flux at 1200 Mc/s measured at the Sydney Radiophysics Laboratory (**7, 8**) and corrected for 1947 by the factor 1·9 (**9**).

6 The daily flux at 600 Mc/s also from the Sydney Radiophysics Laboratory (**7, 8**) and corrected for 1947 *and* 1949 by the factor 1·2 (**9**).

These are the only decimetre-wave data available over an appreciable part of the sunspot cycle. Monthly means have been derived.

The solar activity data used were:

R Daily and monthly Zürich sunspot numbers (**7, 10**).

A Daily and monthly corrected sunspot areas averaged from Washington (**11**) and Greenwich (**12, 13, 14**).

F Daily and monthly corrected facular areas from Greenwich (**12, 13, 14**). Since faculae are seen only near the limb, the daily values were converted to the Sun's centre by adding the eastern faculae of 5 days earlier to the western faculae of 5 days later. Data were supplied by the Greenwich Solar Department for this purpose.

The reason for using three different measures of solar activity was that the complications (*a*) to (*g*) take on a somewhat different form in each case while the deduced segregation of the quiet and slowly varying radiation should remain the same. The repetition of the analysis for three different activity variables provides a much needed check on the results.

As far as possible the six sets of data, which we will label *28*, *12*, *6*, *R*, *A*, *F*, were treated alike.

In order to examine the material the values were plotted out on daily and monthly charts. The 1951 daily chart is shown in Fig. 2, and the monthly chart in Fig. 3. The monthly means were subject to a 13-month smoothing (with

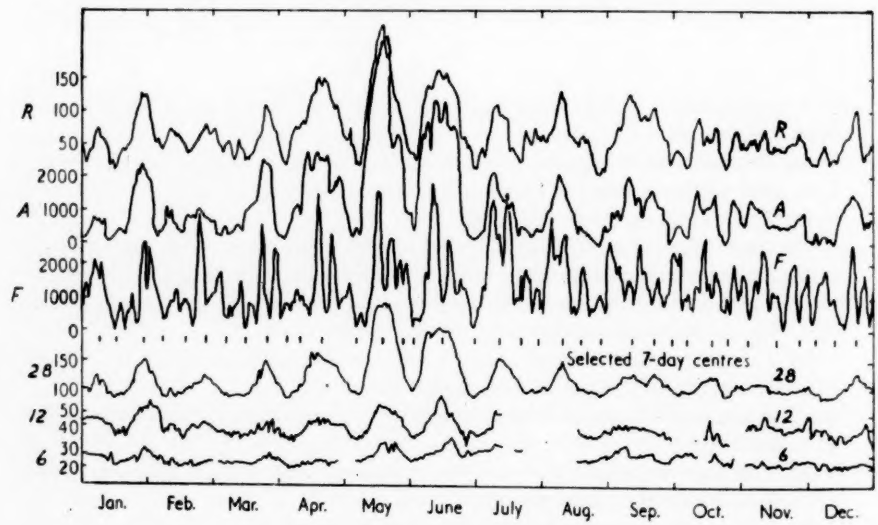

FIG. 2.—*Daily variations during* 1951.

half weight for the extreme months as adopted for Zürich sunspot number smoothing) and the values correlated were the departures from the smoothed curve.

In order to minimize the regression ambiguity (*a*) it was desirable to have a good correlation and a wide range. Monthly values were considered, but their range is too small for these to be suitable for the main correlation. Daily values are better but they require some averaging to cut down the scatter and to keep the numerical work within reasonable bounds. The daily plots (Fig. 2) show that the maxima and minima (which are to supply the key points for the

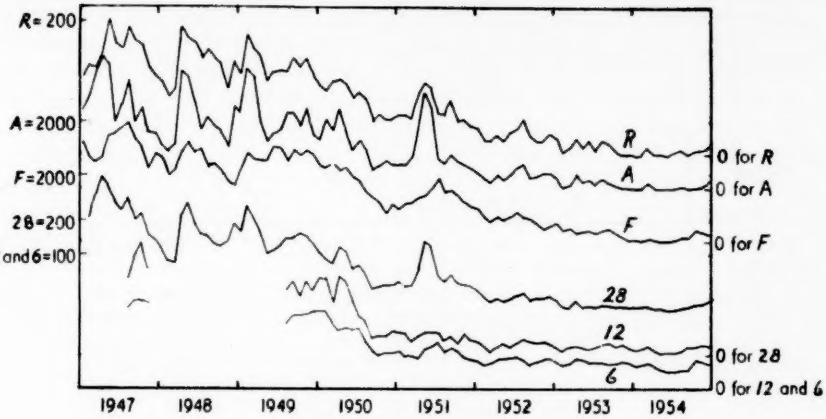

FIG. 3.—*Monthly variations.*

correlation) last about 7 days. Consequently weekly means were used for the main data. We could not afford to miss the main maxima and minima by taking random weeks and instead three weeks per month were carefully selected with the view to giving the most useful values for correlation—usually one maximum, one minimum, and one plateau in each month. The central days for these 7-day periods (Fig. 2) were chosen by considering together the daily charts for sunspot area, sunspot number, and 2800 Mc/s radiation. Care was taken to ensure that the sunspot and radiation data were given equal consideration in selecting the days. The weekly values for the correlation were scaled from the daily charts by using a mask to obscure all but the 7 days concerned and then measuring the mean distance of the daily curve from the 13-month smoothed curve.

The departures from the smoothed curve so found are designated $S(28)$, $S(12)$, $S(6)$, $X(R)$, $X(A)$, and $X(F)$. They are zero in the mean and therefore in a convenient form for correlating.

The various values of S were plotted and correlated with the simultaneous values of X. The main requirement is the slope dS/dX, since, as can be seen from Fig. 1, the mean quiet radiation in any year is obtained from the mean total radiation minus the product of the slope and the mean activity. The procedure has been to make a primary estimate of the slope and then allow for the corrections to it later. The primary estimate was made both by plotting the values and by considering the two regression slopes $[S^2]/[SX]$ and $[SX]/[X^2]$, where the square brackets signify summation. The study of the plotted values was given most weight since in this way one could discard some badly correlated points on the positive side of the diagram. One can see from Fig. 1 that it is the negative values of X that are to be extrapolated for determining the quiet radiation. When using the regressions the slope estimate was placed between the regressions and nearer to the regression *on* that variable thought to be more accurately represented. Details of this are given in Section 3. Both weekly and monthly values were considered for estimating the slope. The various estimates were weighted as thought appropriate and a mean slope (usually for each year) obtained. Results are given in Table I.

TABLE I

Primary estimates of slope dS/dX
Unit of $S = 10^{-22}$ wm^{-2} (c/s)$^{-1}$
Unit of $X = 1$ sunspot number (for R)
Unit of $X = 1$ millionth of hemisphere (for A and F)

Period	28 on R	28 on A	28 on F	12 on R	12 on A	12 on F	6 on R	6 on A	6 on F
1947	0·87	0·033	0·053	0·30	0·021		0·14	0·010	
1948	0·73	0·037	0·052						
1949	0·72	0·033	0·060	0·19	0·011		0·07	0·003	
1950	0·66	0·031	0·048	0·20	0·009		0·08	0·004	
1951	0·75	0·030	0·046	0·18	0·005		0·07	0·003	
1952	0·50	0·033	0·037	0·16	0·007		0·09	0·004	
1953	0·44	0·030	0·030	0·08	0·004		0·11	0·006	
1954	0·41	0·026	0·025	0·09	...		0·10	...	
1947–49	0·77	0·0343	0·056	0·25	0·014	0·014	0·11	0·006	0·007
1950–51	0·72	0·0306	0·046	0·19	0·007	0·011	0·08	0·004	0·007
1952–54	0·48	0·0320	0·035	0·13	0·006	0·012	0·09	0·005	0·006
1947–54	0·74	0·0326	0·051	0·21	0·008	0·012	0·09	0·0045	0·007

The correlation coefficients $r = [SX]/([S^2][X^2])^{1/2}$ are given in Table II.

TABLE II

Correlation coefficients r for weekly values of S and X

Period	28 with R	28 with A	28 with F	12 with R	12 with A	12 with F	6 with R	6 with A	6 with F
1947–49	0·88	0·93	0·62	0·82	0·69	0·42	0·68	0·60	0·62
1950–51	0·94	0·94	0·71	0·62	0·60	0·50	0·46	0·45	0·40
1952–54	0·77	0·83	0·61	0·56	0·58	0·49	0·37	0·38	0·28
1947–54	0·88	0·92	0·63	0·69	0·62	0·46	0·51	0·45	0·44

3. *Complications and corrections.*—The complications will now be considered in the order mentioned in Section 1. Where possible, correction factors are derived which must be multiplied by the primary slope estimate before it can be used as in Fig. 1 for finding the quiet radiation. The factors are collected in Table V.

(*a*) Most of the scatter in the S, X correlations is associated with the imperfect relation between the radiation and the measures of solar activity. For the correlation of $S(28)$ with $X(R)$ and $X(A)$ the scatter is small (Table II) and it is reasonable to adopt a primary slope (which we later identify with m_r) midway between the regressions, namely $([S^2]/[X^2])^{1/2}$. This implies that the random effects causing the scatter are associated equally with the radiation and the activity. The error factor from this assumption must be less than $1-r$ and is not very serious. Nevertheless there is no way of overcoming this residual ambiguity and it was for this reason that the data were selected in such a way

as to correlate as well as possible. When $S(28)$ is correlated with $X(F)$ the scatter is much greater and we may blame the extra scatter onto the random variations in $X(F)$. Hence for $S(28)/X(F)$ it is appropriate to adopt a primary slope close to the regression on S, i.e. $[S^2]/[SX]$. By a similar argument, since $S(12)$ and $S(6)$ correlate rather poorly with $X(R)$ and $X(A)$, it is evident that the blame this time lies with S and we adapt the slope close to the regression on X. The correlation of $S(12)$ and $S(6)$ with $X(F)$ is also poor. It is thought that the random scatter is associated rather more with S than X in this case and the solution is taken somewhat closer to the regression on X. Not much reliance can be put on the $S(12, 6)/X(F)$ slopes.

(*b*) In order to determine whether the solution should be curved, one may compare the slopes at different stages of the sunspot cycle. The slopes from Table I have been plotted against activity in Fig. 4 and smoothed relations drawn. In general the slope is steeper for greater activity. The correction factor (*b*)

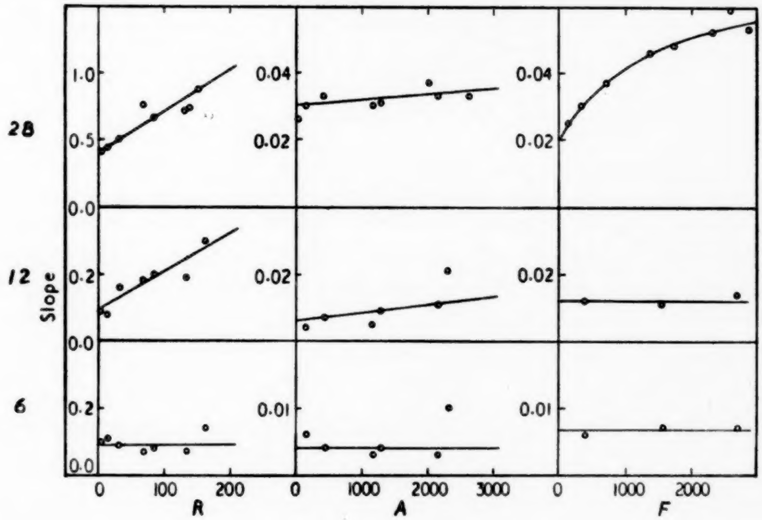

Fig. 4

resulting from the curvature of S on X is simply the area under the slope curve (Fig. 4) to the left of the activity abscissa divided by the product of the ordinate and the abscissa.

It should be mentioned that one would not expect a severe departure from linearity. The characteristics of sunspot groups do not change noticeably with the sp-cycle and the radiations from the groups must be additive. Therefore one would expect both the radiation and the activity to be proportional to the number of spot groups.

(*c*) Changes in the measurement constants could have a serious effect on the present analysis which depends a great deal on differences from one year to the next. There is no reason to mistrust the constants of the activity data which have been given in the same form for many years. The correlations lead to the impression that the 2800 Mc/s data is consistent throughout the period. The correlations of 1200 and 600 Mc/s data with activity are not very good but one

cannot conclude that this is due to undetected changes in the measurement constants. Fortunately the change of this radiation with the sp-cycle is so great that the main conclusions are not likely to be affected by errors of measurement.

(*d*) The variations due purely to the sp-cycle phase change are effectively eliminated by making the measurements from the 13-month smoothed curve.

(*e*) Suppose Fig. 5 represents a characteristic difference in the daily curves of X and S such as might be due to the radiation having a higher directivity from the Sun's centre than the activity. The slope dS/dX would be affected if

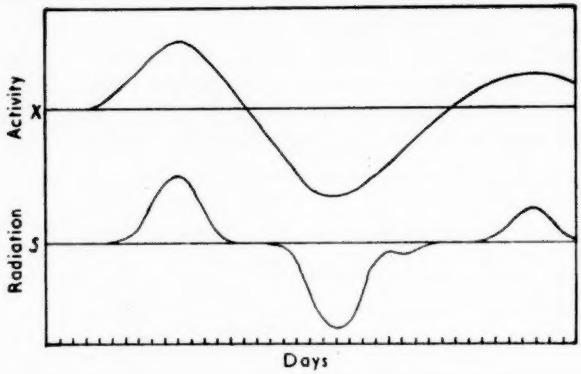

Fig. 5

peak days were chosen (as we have done) whereas if random days had been chosen any peaking effect would have averaged out. A study of the daily charts does not reveal any such characteristic differences. To test this, for each selected peak and dip of $S(28, 12, 6)$ and $X(R, A, F)$ I have measured the value on the central day and also the mean of the values three days earlier and later. Averaging the results gave the ratios of Table III. There does not appear to be any significant change with the sp-cycle, nor do the peaks appear to differ systematically

Table III

Ratio of departure from the smoothed curve three days before and after the selected day to the value on the selected day

	28	*12*	*6*	*R*	*A*	*F*	All
Peaks							
1947–49	0·62	0·62	0·54	0·62	0·63	0·49	0·60
1950–51	0·63	0·55	1·0	0·58	0·85	1·0	0·82
1952–54	0·62	0·76	1·0	0·67	0·68	0·53	0·65
1947–54	0·62	0·60	0·98	0·61	0·69	0·70	0·66
Dips							
1947–49	0·62	0·82	0·50	0·55	0·59	0·67	0·60
1950–51	0·71	0·68	0·73	0·60	0·67	0·77	0·69
1952–54	0·80	1·0	1·0	0·71	0·61	0·50	0·67
1947–54	0·67	0·79	0·70	0·59	0·62	0·67	0·63
Mean	0·65	0·73	0·79	0·60	0·64	0·68	0·64
Increase	1·06	1·00	0·95	1·10	1·06	1·04	1·06

from the dips. To obtain a mean, double weight was given to the dips since these are more important to the analysis. The smaller this ratio the more the measurement has been increased by the policy of selecting the peaks. The amount of this increase, given in the last line of Table III, is obtained by considering typical curves such as shown in Fig. 5. The correction (e) to allow for it is the (activity)/(radiation) ratio of these increases.

(f) See Section 6.

(g) The effect of any time difference in the radio emission and the optically observable activity is one of the main interests of this work and is discussed in Section 4 where a formula for the correction factor (g) is derived.

4. *Persistence and delay of activity or radiation.*—The investigations of Piddington and Davies (**4**), of Waldmeier (**5**) and of Dodson (**15**) suggest that the radiations from sunspot areas disappear less rapidly than the sunspots themselves, and these authors consider the delayed radiation from old sunspots to be the main cause of variation in the non-sunspot component. However Piddington and Davies' composite areas (**4**, p. 588) imply a delay of radiation variation of at least half a month whereas a study of Fig. 3 for 2800 Mc/s radiation reveals almost no delay. Piddington and Davies gave no method of determining the delay quantitatively but showed the importance of controlling the delayed data by comparing it with advanced data. Waldmeier made a quantitative estimate of radiation delayed by one rotation but did not prove the reality of the effect by comparing it with radiation advanced by one rotation. Had he done so he would probably have found that much of his effect was not due to delay. The present work is rather similar to these in aim and approach. However the factors determined are quantitative, and allowance is made for the persistence of both the activity and the radiation, which means that advanced and delayed data are compared on the same basis.

The analysis depends on the correlations of $S(28, 12, 6)$ and $X(R, A, F)$ with one another, both for simultaneous values and for values separated in time by a small number of 27-day rotations. We will use the notation $X_{-1}, X_{-2}, \ldots$ to represent the value of X at 1, 2, ... rotations earlier than the occasion being considered. S and X are taken as the departures of the radiation or activity from the 13-month smoothed mean, and hence the summations such as $[S]$, $[X]$, and their uncorrelated products, will be approximately zero.

The amount of solar activity X at a particular time will be made up of new activity Y plus the remains of activity that was new 1, 2, ... rotations earlier. We can then write

$$\left.\begin{aligned} X_{-1} &= Y_{-1} + aY_{-2} + bY_{-3} + \\ X &= Y \quad + aY_{-1} + bY_{-2} + \\ X_{+1} &= Y_{+1} + aY \quad + bY_{-1} + \end{aligned}\right\}, \tag{1}$$

where $a, b, c, \ldots$ are the factors by which the activity will be decreased in 1, 2, 3, ... rotations. We will assume that there is no correlation between $\ldots Y_{+1}, Y, Y_{-1}, Y_{-2}, \ldots$. Multiplying and adding (e.g. the weekly values throughout a year) gives

$$\left.\begin{aligned} [X^2] &= [Y^2](1 + a^2 + b^2 + \ldots) \\ [XX_{-1}] = [XX_{+1}] &= [Y^2](a + ab + bc + \ldots) \\ [XX_{-2}] = [XX_{+2}] &= [Y^2](b + ac + bd + \ldots) \end{aligned}\right\}, \tag{2}$$

and the low order auto-correlation coefficients for adjacent and nearby rotation months become

$$\left.\begin{aligned}[XX_{-1}]/[X^2] &= (a+ab+bc+\ldots)/(1+a^2+b^2+\ldots)\\ [XX_{-2}]/[X^2] &= (b+ac+bd+\ldots)/(1+a^2+b^2+\ldots)\end{aligned}\right\}. \tag{3}$$

Now take the radiation S emitted from the activity Y to be mY, and the factor by which S varies in subsequent rotations to be α, β, γ, ..., then

$$S = m(Y+\alpha Y_{-1}+\beta Y_{-2}+\ldots). \tag{4}$$

The S auto-correlations are similar to (3) with α, β, ... replacing a, b, ... By multiplying the items of (4) with (1) and adding

$$\left.\begin{aligned}[SX] &= m[Y^2](1+a\alpha+b\beta+\ldots)\\ [SX_{-1}] &= m[Y^2](\alpha+a\beta+b\gamma+\ldots)\\ [SX_{-2}] &= m[Y^2](\beta+a\gamma+b\delta+\ldots)\\ [SX_{+1}] &= m[Y^2](a+\alpha b+\beta c+\ldots)\\ [SX_{+2}] &= m[Y^2](b+\alpha c+\beta d+\ldots)\end{aligned}\right\}. \tag{5}$$

The regression slopes become

$$\left.\begin{aligned}S \text{ on } X &= [SX]/[X^2] = m(1+a\alpha+b\beta+\ldots)/(1+a^2+b^2+\ldots)\\ X \text{ on } S &= [S^2]/[SX] = m(1+\alpha^2+\beta^2+\ldots)/(1+a\alpha+b\beta+\ldots)\end{aligned}\right\}, \tag{6}$$

and the mean of these is

$$m_r = ([S^2]/[X^2])^{1/2} = m(1+\alpha^2+\beta^2+\ldots)^{1/2}/(1+a^2+b^2+\ldots)^{1/2}. \tag{7}$$

It must be noticed that the scatter entering these statistics is associated with the non-correlation of Y with Y_{-1}, Y_{-2}, ... only. Other random effects such as errors of measurements and variability of m are not included. However m_r *can* be identified with the primary estimate of slope mentioned in Section 2.

The summations of (5) are mainly used in the form of ratios such as

$$[SX_{-1}]/[SX] = (\alpha+a\beta+b\gamma+\ldots)/(1+a\alpha+b\beta+\ldots), \text{ etc.,} \tag{8}$$

which are numerical ratios of various regression slopes and eliminate m and $[Y^2]$.

We now consider our main purpose which is to find the radiation for zero activity, i.e. to find S when $X=-M$, M being the smoothed sunspot activity. For a completely quiet Sun Y, Y_{-1}, Y_{-2}, ... must all represent zero activity and will all have the same negative value Y_{-M}. Then from (1)

$$Y_{-M} = -M/(1+a+b+\ldots).$$

The corresponding zero activity value of S is, from (4) and (7),

$$\begin{aligned}S_{-M} &= -Mm(1+\alpha+\beta+\ldots)/(1+a+b+\ldots)\\ &= -Mm_r(g),\end{aligned} \tag{9}$$

where

$$(g) = \frac{1+\alpha+\beta+\ldots}{1+a+b+\ldots}\,\frac{(1+a^2+b^2+\ldots)^{1/2}}{(1+\alpha^2+\beta^2+\ldots)^{1/2}}. \tag{10}$$

Thus (g) is the factor by which the primary slope estimate m_r must be multiplied to allow for delayed action.

Since the higher members of the series $1, a, b, \ldots, 1, \alpha, \beta, \ldots$ cannot be determined from the correlations with adequate accuracy it is necessary to seek simpler series to replace them. The most satisfactory solution is

$$\left.\begin{array}{l} 1, a, b, c, \ldots \rightarrow 1, a, a^2, a^3, \ldots \\ 1, \alpha, \beta, \gamma, \ldots \rightarrow 1, \alpha, \alpha^2, \alpha^3, \ldots \end{array}\right\}. \tag{11}$$

The relevant formulae (3) and (8) then become

$$\left.\begin{array}{l} [XX_{-1}]/[X^2] = a = [SX_{+1}]/[SX] \\ [XX_{-2}]/[X^2] = a^2 = [SX_{+2}]/[SX] \\ [SS_{-1}]/[S^2] = \alpha = [SX_{-1}]/[SX] \\ [SS_{-2}]/[S^2] = \alpha^2 = [SX_{-2}]/[SX] \end{array}\right\}, \tag{12}$$

and (10) becomes

$$(g) = \left(\frac{1-a}{1-\alpha} \cdot \frac{1+\alpha}{1+a}\right)^{1/2}. \tag{13}$$

We see from (12) that both the auto-correlations (left side) and the regression slope ratios (right side) give values of a and α.

The values of $[XX_{-1}]$, $[X^2]$, $[SX_{-1}]$, etc., obtained from the data are not precisely the same as the summations required for (3), (8) and (12) since the former are increased by some random scatter. This extra scatter will not appreciably affect the ratios in (8) and the right side of (12) since they can be regarded as ratios of regressions *on* the same variable. However in (3) and the left side of (12) the $[X^2]$ and $[S^2]$ from the data will be artificially too high and we must be prepared to increase the ratios by a few per cent.

It is important also to allow for the fact that the selection of weeks was not a random one. For this reason $[X^2]$ was somewhat greater than would be obtained from a normal dispersion and greater than $[X^2_{-1}]$ or $[X^2_{-2}]$. An allowance for such differences has been made by making all the correlations and ratios dimensionless, e.g. using $[XX_{-1}]/([X^2][X^2_{-1}])^{1/2}$ on the left of (3) or in (12), and $[SX_{-1}][X^2]^{1/2}/[SX][X^2_{-1}]^{1/2}$ on the left of (8) or in (12).

5. *Results.*—The results that might be expected to come from the analysis are (i) the decay factors represented by a and α, (ii) the flux of slowly varying radiation, and (iii) the flux of quiet radiation.

(i) *The decay rates.*—The values of $a(R)$, $a(A)$ and $\alpha(28)$ obtained from the various parts of (12) are in reasonable agreement and suggest that (11) is applicable for R, A, and *28*. For F, *12*, and *6* the decay is slower and disagreements in (12) show that (11) is not entirely adequate. Nevertheless the data are not consistent enough to justify the attempt to determine more than one constant for each phenomenon and the results have been averaged from (12) as well as possible. Monthly data have been analysed for this purpose as well as weekly, and the decay constants finally adopted are given in Table IV. The values of a and α refer to 27-day periods. For comparison with other data the decays have also been converted into mean-life in calendar months.

The variations in the mean-life of solar phenomena have been demonstrated in the past (**16, 17**) by correlation delays. It can be seen from (12) that the correlation delay will be approximately half the difference of mean-life for the two phenomena concerned. Values are given in Table IV. The delays with respect to R have been determined also by the earlier method (**16**) and these too are given in Table IV. The comparison is reasonably satisfactory.

TABLE IV

Values are averaged to represent the whole period 1947–54

	R	*A*	*F*	*28*	*12*	*6*
a or α (27-day)	0·59	0·53	0·72	0·59	0·67	0·70
Mean life (in months)	1·7	1·4	2·7	1·7	2·2	2·5
Half mean-life difference	0·0	−0·15	+0·50	0·0	+0·25	+0·40
Delay from *R* (in months)	0·0	−0·4	+0·6	−0·2	+0·1	+0·4

Points of interest in Table IV are as follows. There is an increase of mean-life and delay as the frequency is decreased. At 2800 Mc/s there is no significant delay as compared with sunspot activity. The longer life and increased delay of faculae in comparison with sunspots has long been known (**16**). The difference between *R* and *A* is quite surprising. It is apparently due to an effect whereby the number of spot groups on the Sun tends to increase one or two rotations after, and in the neighbourhood of, an enormous sunspot. This result appears clearly for the six longest sunspots ever observed (**14**), half of which come within the period studied.

(ii) *Slowly varying radiation.*—The various yearly means and factors from which the slowly varying radiation has been determined are assembled in Table V. The units are Zürich numbers for $\bar{R}$, millionths of the visible hemisphere for $\bar{A}$ and $\bar{F}$, and 10^{-22} wm^{-2} (c/s)$^{-1}$ for radiation; corresponding units are used for the slopes. The correction factors (*b*), (*e*), and (*g*) have been explained in Sections 3 and 4. A certain amount of smoothing of the data has been adopted; in many cases the value quoted is the mean of the individual and the smoothed value. The slowly varying component of radiation in the last column is the product of the five columns preceding it.

The interagreement of the values in the last column obtained almost independently from *R*, *A*, and *F* should give an indication of the reliability of the procedures used. Table VI gives the weighted mean of the same values—the probable errors should be about 1 or 2 units in this table. The fluxes associated with various measures of activity (each representing approximately a normal sunspot maximum) are given in Table VII. The slowly varying flux is proportional to sunspot area but the relation with sunspot number and faculae is not quite linear.

The flux is seen to be approximately proportional to the radiation frequency in the range studied. However this relation does not persist for frequencies above 2800 Mc/s. At 3750 Mc/s Nagoya observations (**7**) show rather less flux for a specified sunspot area than at 2800 Mc/s, and at 9400 Mc/s Sydney observations (**7**) show barely detectable slowly varying radiation—perhaps about 1/10 the flux at 2800 Mc/s. These points have been clarified by a graph plotted out by S. F. Smerd and kindly sent to the author.

(iii) *Quiet Sun radiation.*—The flux from the quiet Sun remains after the slowly varying component has been substracted from the total. The yearly values are shown in Table VI and the values associated with defined measures of activity are given in Table VII in terms of both flux and apparent solar temperature.

TABLE V

Radiation Activity	Year	$\bar{R}$ $\bar{A}$ $\bar{F}$	Slope	(*b*)	(*e*)	(*g*)	Slowly varying radiation 10^{-22} wm^{-2} (c/s)$^{-1}$
28 from *R*	1947	152	0·87	0·73	1·03	0·97	97
	48	136	0·78	0·74	,,	0·97	78
	49	135	0·77	0·75	,,	0·97	77
	50	84	0·67	0·80	,,	1·02	47
	51	69	0·68	0·83	,,	1·02	41
	52	32	0·50	0·90	,,	1·02	15
	53	14	0·44	0·94	,,	1·02	6
	54	4	0·41	0·98	,,	1·02	2
28 from *A*	1947	2640	0·0336	0·95	1·00	1·08	91
	48	2010	0·0350	0·96	,,	1·08	73
	49	2160	0·0332	0·96	,,	1·08	75
	50	1300	0·0314	0·97	,,	1·11	44
	51	1170	0·0312	0·98	,,	1·11	40
	52	430	0·0317	0·99	,,	1·06	14
	53	150	0·0302	1·00	,,	1·06	5
	54	40	0·0281	1·00	,,	1·06	1
28 from *F*	1947	2890	0·055	0·78	0·98	0·80	97
	48	2330	0·052	0·78	,,	,,	74
	49	2600	0·057	0·78	,,	,,	90
	50	1750	0·049	0·78	,,	,,	52
	51	1380	0·046	0·77	,,	,,	38
	52	710	0·037	0·80	,,	,,	16
	53	330	0·030	0·83	,,	,,	6
	54	140	0·025	0·90	,,	,,	2
12 from *R*	1947	162	0·28	0·68	1·10	1·14	39
	49	133	0·21	0·70	,,	,,	25
	50	84	0·20	0·76	,,	,,	16
	51	69	0·18	0·78	,,	,,	12
	52	32	0·15	0·87	,,	,,	6
	53	14	0·10	0·94	,,	,,	1·7
	54	4	0·10	0·98	,,	,,	0·5
12 from *A*	1947	2320	0·017	0·78	1·06	1·25	41
	49	2070	0·011	0·79	,,	,,	25
	50	1300	0·009	0·83	,,	,,	13
	51	1170	0·007	0·84	,,	,,	9
	52	430	0·007	0·94	,,	,,	4
	53	150	0·005	0·98	,,	,,	1·0
	54	40	0·006	0·99	,,	,,	0·3
12 from *F*	1947	2940	0·012	1·00	1·04	0·91	34
	49	2650	,,	,,	,,	,,	30
	50	1750	,,	,,	,,	,,	20
	51	1380	,,	,,	,,	,,	16
	52	710	,,	,,	,,	,,	8
	53	330	,,	,,	,,	,,	4
	54	140	,,	,,	,,	,,	1·6
6 from *R*	1947	162	0·12	1·00	1·16	1·23	28
	49	133	0·08	,,	,,	,,	16
	50	84	0·08	,,	,,	,,	10
	51	69	0·08	,,	,,	,,	8
	52	32	0·09	,,	,,	,,	4
	53	14	0·10	,,	,,	,,	2
	54	4	0·09	,,	,,	,,	0·5
6 from *A*	1947	2320	0·008	1·00	1·12	1·32	28
	49	2070	0·004	,,	,,	,,	12
	50	1300	0·004	,,	,,	,,	8
	51	1170	0·004	,,	,,	,,	7
	52	430	0·004	,,	,,	,,	2·7
	53	150	0·005	,,	,,	,,	1·1
	54	40	0·004	,,	,,	,,	0·2
6 from *F*	1947	2940	0·007	1·00	1·10	0·96	22
	49	2650	,,	,,	,,	,,	19
	50	1750	,,	,,	,,	,,	13
	51	1380	,,	,,	,,	,,	10
	52	710	,,	,,	,,	,,	5
	53	330	,,	,,	,,	,,	2·4
	54	140	,,	,,	,,	,,	1·0

TABLE VI

Annual mean values of total radiation, slowly varying radiation, and quiet radiation
Unit$=10^{-22}$ wm^{-2} (c/s)$^{-1}$

Year	Total			Slowly Varying			Quiet		
	28	*12*	*6*	*28*	*12*	*6*	*28*	*12*	*6*
1947	220	96	64	94	39	27	126	57	37
48	176	...	...	75	...	...	101	...	...
49	178	76	55	78	26	15	100	50	40
50	129	61	41	47	16	10	82	45	31
51	121	40	27	40	12	8	81	28	19
52	87	34	21·4	15	6	3·6	72	28	18
53	75	31	20·1	6	2	1·7	69	29	18
54	72	29·5	16·7	2	0·6	0·5	70	29	16

TABLE VII

Radiation associated with specified measures of solar activity

	Radiation	$R=A=F=0$	$R=100$	$A=2000$	$F=2000$
Slowly varying flux	*28*	0	55	72	58
(Unit$=10^{-22}$ wm^{-2} (c/s)$^{-1}$)	*12*	0	20	26	19
	6	0	13	16	13
Quiet flux	*28*	70	90	100	91
(Unit$=10^{-22}$ wm^{-2} (c/s)$^{-1}$)	*12*	29	41	46	42
	6	17	27	32	28
Quiet Sun apparent	*28*	43	55	61	55
temperature (Unit$=1000^\circ$K)	*12*	96	137	153	140
	6	230	360	425	370

At all three frequencies there is an increase of quiet radiation with increased solar activity. At each frequency the flux varies approximately quadratically with the solar activity in accordance with the empirical relations (in units 10^{-22} wm^{-2} (c/s)$^{-1}$):

$$\text{Quiet flux } 28 \simeq 70+20(R/100)^2 \simeq 70+7{\cdot}5(A/1000)^2,$$
$$12 \simeq 29+12(R/100)^2 \simeq 29+4{\cdot}3(A/1000)^2,$$
$$6 \simeq 17+10(R/100)^2 \simeq 17+3{\cdot}8(A/1000)^2.$$

The accuracy is not sufficient to establish the quadratic form of this variation but the change between sunspot maximum and minimum is much greater than any likely error. The relative change of flux is systematically greater for smaller frequencies. At sp-min the flux is approximately proportional to the frequency (as already found for the slowly varying component) but this does not apply at sp-max.

In Fig. 6 the quiet flux values (expressed as apparent temperature of the Sun's disk) for $A=0$ and $A=2000$ are compared with a recent survey (**18**). $A=2000$ can be regarded as representing the recent sp-max activity. The new

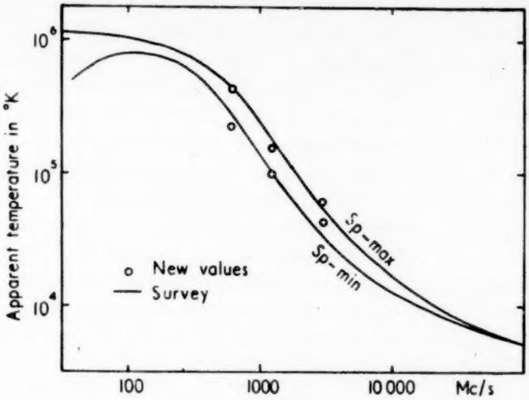

FIG. 6.—*New values of apparent solar temperature compared with a recent survey* (**17**).

measurements call for no great change in the previously published curves but suggest that the two curves sp-max and sp-min instead of being parallel should be closer at the high frequency end and wider at low frequencies.

6. *Conclusion.*—The main conclusion coming from the analysis is that the variation of quiet Sun decimetre radiation is small but real. This conclusion differs from others recently published (**4, 5, 15**).

It is important to enquire whether the analysis has omitted any factor that could influence this result and we are reminded that the complication (*f*) of Section 1 is not yet resolved. Are there any local sources uncorrelated with our *R*, *A* or *F*? We consider first whether uncorrelated courses could be expected physically, and secondly whether there is any evidence of them.

The local sources appear to be hot regions, i.e. regions where the high coronal temperatures have penetrated into the higher density material (**19, 20**). There is every reason to think that faculae represent these hot regions in the high photosphere. We would expect the emission sources to be closely connected with faculae and therefore with the other phenomena that show an almost one to one correspondence with faculae such as calcium plages, regions of magnetic field, regions of 5303 emission, and sunspot groups. The only difference between these from the statistical or correlation point of view concerns the delay and mean life for which adequate allowance has been made. If the emission is a thermal effect we would not expect it from sources which did not show the similar thermal evidence. Therefore we would *not* expect any local sources that are uncorrelated with sunspots or faculae. If the same analysis could be carried out using plage brightness or area of high 5303 coronal emission it should give the same result again.

Turning to the evidence from observation we find that nearly all the observed local sources of emission can be identified (**21, 22, 23, 24, 25**) with sunspot groups, plages, or 5303 regions. However one of the local emissions at the eclipse of 1948 November 1 (**21**) was identified with a stable prominence. Stable prominences (seen as filaments on the disk) are practically uncorrelated with the solar activity data we have used. Numerical data on prominences from Catania, Tokyo, Madrid, and Kodaikanal in the period 1947–54 have been examined

with the view to seeking a prominence component in the slowly varying radiation. Unfortunately the prominence measurements do not appear to be consistent enough to use them for this purpose. There was only one clear feature that appeared on all the data and was not seriously confused with other activity. This was a maximum of prominences in 1951 September–October. From Fig. 3 we see no corresponding increase in *28* or *12* but some evidence of an effect in *6* (the frequency of the 1948 eclipse result). We might conclude that if there is a prominence effect it increases with decreasing frequency.

These arguments are somewhat contradictory. It cannot be decided with certainty whether uncorrelated local sources (i.e. sources that are not associated with centres of sunspot activity) exist or whether they produce sufficient radiation to affect the issue. The author's personal opinion is that the total emission from such sources is negligible and that the data of Tables VI and VII can be used directly for quiet Sun problems.

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THE 2^3S-2^3P, 3^3P, 3^3D, 4^3D EXCITATIONS OF HELIUM ATOMS BY ELECTRONS

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Summary

Born's approximation is used to calculate the cross sections for the excitation of the 2^3P, 3^3P, 3^3D and 4^3D states of helium from the metastable 2^3S state by electrons with energies ranging from the thresholds of the processes to about 100 eV. In the case of the 2^3S-2^3P excitation a close coupling formula due to Seaton is also used.

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1. *Introduction.*—Goldberg (**1**) has investigated the relative intensities of the helium lines for a nebula in radiative equilibrium, making no allowance for the effects of collisions with electrons. The lack of accord between theory and observation which was obtained might well be due to the neglect of collisions (**2**). Unfortunately it has not yet been possible to take them into account, due to the lack of knowledge of the relevant cross sections. It seems desirable, therefore, to carry out calculations on the collision processes which are of importance in gaseous nebulae. Those involving the $2\,^3S$ metastable state of helium are of particular interest in this connection.

In this paper, Born's approximation has been used to obtain the cross sections for the $2\,^3S-2\,^3P$, $3\,^3P$, $3\,^3D$, $4\,^3D$ excitations of helium atoms by electrons. The energy range investigated in these calculations is from the thresholds of the processes to about 100 eV. Born's approximation tends to overestimate the cross sections, especially when the coupling between the levels involved is strong. As this occurs for the $2\,^3S-2\,^3P$ excitation, a close coupling formula based on Bethe's approximation which is due to Seaton (**3**) was also used in this particular case.

2. *Theory.*—The Born approximation to the cross section for the excitation of the $1snl\,^3L$ state of helium from the metastable $1s2s\,^3S$ state is given by the expression

$$Q(2s\,^3S-nl\,^3L)=\frac{8\pi^3m^2}{k_i^2h^4}\int_{K_{\min}}^{K_{\max}}|\mathcal{N}(2s\,^3S-nl\,^3L)|^2K\,dK \tag{1}$$

where

$$\mathcal{N}(2s\,^3S-nl\,^3L)=\iiint\phi(2s\,^3S;\,\mathbf{r}_1,\mathbf{r}_2)\phi(nl\,^3L;\,\mathbf{r}_1,\mathbf{r}_2)V(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)\exp i\mathbf{K}.\mathbf{r}_3\,d\mathbf{r}_1\,d\mathbf{r}_2\,d\mathbf{r}_3 \tag{2}$$

and

$$\mathbf{K}=\mathbf{k}_i-\mathbf{k}_f,\quad \mathbf{k}_i=2\pi m\mathbf{v}_i/h,\quad \mathbf{k}_f=2\pi m\mathbf{v}_f/h; \tag{3}$$

$\mathbf{v}_i$ and $\mathbf{v}_f$ are the initial and final velocities of the incident electron; $\mathbf{r}_1$ and $\mathbf{r}_2$ are the position vectors of the bound electrons relative to the helium nucleus and $\mathbf{r}_3$ that of the the free electron; the ϕ's are the wave functions of the helium atom

in the states indicated and V is the interaction potential which is given by

$$V(\mathbf{r}_1, \mathbf{r}_2; \mathbf{r}_3) = \frac{e^2}{r_{13}} + \frac{e^2}{r_{23}} - \frac{2e^2}{r_3}. \tag{4}$$

If we write the helium wave functions in the form

$$\phi(nl\,^3\mathrm{L}; \mathbf{r}_1, \mathbf{r}_2) = \frac{1}{\sqrt{2}} \{u(1s; \mathbf{r}_1)u(nl; \mathbf{r}_2) - u(1s; \mathbf{r}_2)u(nl; \mathbf{r}_1)\} \tag{5}$$

where the atomic orbitals u are normalized and mutually orthogonal, we obtain

$$\mathcal{N}(2s\,^3\mathrm{S} - nl\,^3\mathrm{L}) = \frac{4\pi a_0^2 e^2}{t^2} |\mathcal{I}(2s - nl)| \tag{6}$$

where

$$\mathcal{I}(2s - nl) = \int u(2s; \mathbf{r})u(nl; \mathbf{r}) \exp i\mathbf{t}.\mathbf{r} d\mathbf{r} \tag{7}$$

and

$$\mathbf{t} = \mathbf{K}a_0, \tag{8}$$

$\mathbf{r}$ being in units of a_0, the radius of the first Bohr orbit of hydrogen.

For the $2\,^3\mathrm{S} - 2\,^3\mathrm{P}$, $3\,^3\mathrm{D}$ and $4\,^3\mathrm{D}$ excitations, the helium 2s atomic orbital was chosen to have the form

$$u(2s; \mathbf{r}) = \sqrt{\frac{3\beta^5}{\pi(\alpha^2 - \alpha\beta + \beta^2)}} \{1 - \tfrac{1}{3}(\alpha + \beta)r\} e^{-\beta r} \tag{9}$$

with

$$\alpha = 2{\cdot}00, \quad \beta = 0{\cdot}695$$

while $u(2p; \mathbf{r})$, $u(3d; \mathbf{r})$ and $u(4d; \mathbf{r})$ were assumed to be hydrogenic. The $2\,^3\mathrm{S} - 3\,^3\mathrm{P}$ cross section was found to be very sensitive to the parameters of both the initial and final orbitals. Consequently, for this particular case, use was made of the more accurate atomic orbitals obtained by Morse, Young and Haurwitz (4) and by Goldberg and Clogston (5); they are

$$u(2s; \mathbf{r}) = \frac{1}{\sqrt{4\pi}} M(re^{-\mu r} - Ae^{-ar}) \tag{10}$$

$$u(3p; \mathbf{r}) = \sqrt{\frac{3}{4\pi}} \cos\theta Nr(re^{-\lambda r} - Be^{-br}) \tag{11}$$

with

$$M = 0{\cdot}356, \quad A = 2{\cdot}93, \quad a = 1{\cdot}57, \quad \mu = 0{\cdot}610$$

and

$$N = 0{\cdot}00990, \quad B = 14{\cdot}2, \quad b = 0{\cdot}490, \quad \lambda = 0{\cdot}318.$$

In the case of the $2\,^3\mathrm{S} - 2\,^3\mathrm{P}$ excitation of helium, the value of the line strength S given by the expression

$$\mathrm{S} = 3\left|\int u(2s; \mathbf{r})u(2p; \mathbf{r})\, r\cos\theta\, d\mathbf{r}\right|^2 \tag{12}$$

is 19·3. This implies that the coupling between the initial and final states is strong and hence that Born's approximation must greatly overestimate near the threshold. Seaton (3) has shown that in such cases a more accurate procedure can be based on Bethe's approximation

$$Q_{\mathrm{Bethe}} = \frac{8\pi a_0^2}{3k_i^2} \mathrm{S} \log_e \frac{k_i + k_f}{k_i - k_f}. \tag{13}$$

If the coupling is strong the first few partial cross sections Q^l_{Bethe} will exceed the theoretical maximum

$$Q^l_{\max} = \frac{\pi a_0^2}{k_i^2}(2l + 1)$$

imposed by charge conservation. Suppose l_0 is the value of l for which

$$Q^l_{\mathrm{Bethe}} \geqq Q^l_{\max} \; (l \leqq l_0)$$
$$< Q^l_{\max} \; (l > l_0).$$

Putting $Q^l = \frac{1}{2}Q^l_{\max}$ for $l \leqq l_0$ and $Q^l = Q^l_{\mathrm{Bethe}}$ for $l > l_0$, Seaton obtains

$$Q = \frac{\pi}{k_1^2}\frac{1}{2}(2l_0+1)^2 + Q_{\mathrm{Bethe}} - \sum_{=0}^{l_0} Q^l_{\mathrm{Bethe}}. \tag{14}$$

This formula, as well as Born's approximation, has been used in the present paper to obtain the cross section for the $2\,^3S-2\,^3P$ excitation.

3. *Results.*—Displayed in the table are the values of the cross sections for the $2\,^3S-2\,^3P$, $3\,^3P$, $3\,^3D$ and $4\,^3D$ excitations of helium atoms by electrons with energies ranging from the thresholds of the various processes (1·14 eV, 3·19 eV, 3·25 eV and 3·92 eV respectively) to about 100 eV.

TABLE I

Cross sections Q for the 2^3S-2^3P, 3^3P, 3^3D *and* 4^3D *excitations of helium atoms by electrons*

Electron Energy (in eV.)	$Q(2^3S-2^3P)$ (in πa_0^2) Born	Equation (14)	$Q(2^3S-3^3P)$ (in πa_0^2)	$Q(2^3S-3^3D)$ (in πa_0^2)	$Q(2^3S-4^3D)$ (in πa_0^2)
1·14	0	0			
1·22	173				
1·72		79			
2·18	294	94			
3·40	248	110	1·08	3·60	
4·89	202	114	1·29	6·94	1·69
6·66	167	112	1·13	6·34	1·73
8·70	138	105	1·07	5·41	1·51
13·6	101	91	0·99	3·85	1·09
19·6	76	77	0·91	2·81	0·80
30·6	54	60	0·78	1·87	0·53
54·4	34	41	0·59	1·08	0·31
85·0	24	30	0·46	0·70	0·20
122	18	23	0·37	0·49	0·14

The thresholds for the 2^3S-3^3P, 3^3D and 4^3D excitations occur at 3·19 eV, 3·25 eV, 3·92 eV, respectively.

The effect of using Seaton's modification of Bethe's approximation instead of the Born approximation is to reduce the cross section for the $2\,^3S-2\,^3P$ excitation by nearly one-third close to the threshold and also drastically to alter the shape of the peak. At higher electron energies the two approximations agree quite well.

The cross section for the $2\,^3S-3\,^3P$ excitation is very much smaller than that for the $2\,^3S-2\,^3P$ excitation, their maximum values being about $1{\cdot}3\pi a_0^2$ and $115\pi a_0^2$ respectively. The former cross section is abnormally small, mainly due to the fact that the line strength S for the $2\,^3S-3\,^3P$ transition is only 0·92. Consequently, the $2\,^3S-3\,^3D$ and even the $2\,^3S-4\,^3D$ excitation cross sections are larger than the $2\,^3S-3\,^3P$ cross section near the threshold, having maximum values of about $7\pi a_0^2$ and $1{\cdot}8\pi a_0^2$ respectively.

It is a pleasure to thank Dr. J. W. Chamberlain of Yerkes Observatory for drawing attention to the need for this investigation.

The Queen's University,
Belfast :
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RADIAL VELOCITIES AND SPECTRAL TYPES IN THE GALACTIC CLUSTERS M 25 AND NGC 6087

M. W. Feast

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Summary

67 spectra of 42 stars in M 25 and 38 spectra of 25 stars in NGC 6087 have been used for the determination of spectral types and radial velocities. The majority of the stars observed in each cluster are late B and early A main sequence stars. The radial velocity results support the contention that the classical cepheids U Sgr and S Nor are members of M 25 and NGC 6087 respectively.

Introduction.—The southern galactic clusters M 25 (I.C. 4725) and NGC 6087 are of interest since near each of their centres is a classical cepheid (U Sgr and S Nor, respectively).

Cluster	α (1950)	δ (1950)	l	b	Type (1)	Distance (1)
M 25	18^h29	$-19^\circ10$	341·3	−6·0	2*b*	980 psc
NGC 6087	16^h15	$-57^\circ47$	295·3	−6·3	1–2*b*	870 psc

It is clear that if the cepheids are actually cluster members we have an independent method of determining the zero point of the period–luminosity relation and the clusters may also assume some importance in theories of the evolution of galactic clusters. Dr J. B. Irwin and Dr A. J. Wesselink both drew the writer's attention to the interesting problem of these clusters and, during his recent visits to the Royal Observatory, Cape and the Radcliffe Observatory, Pretoria, Dr Irwin obtained three colour photoelectric observations of the cepheids and of a number of stars in the two clusters. These observations are now being prepared for publication. It was clear that spectroscopic observation of the stars in the two clusters was desirable and it was arranged with Dr Irwin and Dr Wesselink that the writer should carry this out. The purpose of the spectroscopic programme was twofold:

(*a*) To compare the radial velocities of a number of stars in each cluster with the radial velocities of the two cepheids; this is important since, according to Irwin (**2**), the available proper motion data are too poor to decide whether or not the cepheids belong to the clusters.

(*b*) To classify the cluster stars on the MK system for combination with the photometric results in a discussion of the clusters.

Observations.—The observations were made between 1955 July and October using the two-prism spectrograph at the Cassegrain focus of the 74-inch Radcliffe reflector. Either the *f*/3·7 camera (49 A/mm at Hγ) or the *f*/2 camera (86 A/mm at Hγ) were used. In the case of the *f*/2 camera the spectra were considerably widened for spectral classification. The present discussion is based on 67 spectra of 42 stars in M 25 and 38 spectra of 25 stars in NGC 6087.

The majority of the stars observed in the two clusters are main sequence stars of early A or late B type. For these stars the wave-length system recommended by the Victoria workers has been employed with both cameras. Tests of this system on the $f/3{\cdot}7$ camera have been reported previously (**3**). For the $f/2$ camera it is intended to publish results of tests on standard stars later. So far such tests have shown no significant systematic errors between the two cameras. The lines in the AV–BV stars are generally broad and few in number so that it would not be possible to determine accurate velocities for these stars without a much more extensive programme than that reported here. It has therefore been decided to give only the mean velocity of the main sequence stars for each cluster. The programme also included a few late-type stars. Spectra of these with the $f/3{\cdot}7$ camera were measured, as is the normal practice here, using the Victoria wave-lengths (**4**). Those on the $f/2$ camera were measured on a Hartmann spectrocomparator, either γ Hya (gG 5) or HD 119971 (K 5) being used as standards.

The spectral types were determined by comparison with standard stars on the MK system (**5**) and by use of the Yerkes Atlas.

Discussion.—The radial velocity results and their standard errors are given for the two clusters in Table I.

Table I

Radial velocities in M 25

U Sgr	+4 km/sec
54 spectra of 35 stars (BV–AV)	+4 ± 4 km/sec
No. 150 (G5 III)	− 2 ± 5 km/sec
No. 193 (G2 III)	− 1 ± 2 km/sec
No. 26 (K2 V:)	−35 ± 3 km/sec (non-member)
No. 70 (A9 V)	−56 ± 8 km/sec (non-member)

Radial velocities in NGC 6087

S Nor	+ 3 km/sec
27 spectra of 18 stars (BV–AV)	+ 2 ± 3 km/sec
No. x (M1 III)	+ 2 ± 4 km/sec
No. f (G4 III)	−27 ± 2 km/sec (non-member)
No. β (A4 V)	−14 ± 3 km/sec (non-member)
No. t (B9 III:)	−10 ± 7 km/sec

The letters and numbers used to designate the stars in Tables I and II are due to Irwin and will be identified by chart in his forthcoming discussion of the photometry of the clusters.

The normal radial velocities of U Sgr and S Nor are taken from the recent investigation by Stibbs (**6**). According to Stibbs "the estimated average probable error of a single normal velocity is of the order of 1 km/sec". The 35 stars in M 25 and 18 in NGC 6087 used in forming the mean "main sequence" velocities in Table I are indicated by asterisk in Table II. There would appear to be little doubt that they are all cluster members. The agreement of the mean "main sequence" velocities in each case with the velocities of the cepheids strongly supports the contention that the cepheids are indeed cluster members.

Stars other than the BV–AV stars are listed separately in Table I. However, omitted from this table are the M giants 6087/e and 25/49 for which only one

f/2 spectrum each was available. The velocities obtained for these two stars are not of sufficient accuracy to decide on cluster membership. The A type supergiants 25/124 and 25/192 are also omitted from Table I. They are presumably non-members though their radial velocities are too poorly determined to draw definite conclusions from them. The stars 6087/g (B8 V or III) and 6087/v (A0 III : (V ?)) are the only other stars not listed in Table I. It seems likely that they are main sequence cluster members and that their velocities might be incorporated in the "main sequence" mean for NGC 6087 but in view of the uncertainty in their spectral types it has been thought best to omit them.

Turning now to the stars listed separately in Table I, the two clusters may be discussed in turn. In M 25 the G-type giants No. 150 and No. 193 appear from their radial velocities to be cluster members, a conclusion that is not at variance with a rough estimate of their magnitudes compared with those of the main sequence cluster members. On the other hand the stars No. 26 (K2 V :) and No. 70 (A9 V) whose velocities indicate that they are non-members appear in fact to be too bright for membership and to be foreground objects. In NGC 6087 star No. x (M1 III) appears to be a member from its radial velocity whilst No. f (G4 III) seems to be a non-member. This latter result is confirmed by three *f*/2 and one *f*/3·7 spectra. 6087/β (A4 V) was not included in the main sequence mean as it is clearly a non-member. It lies on the outer edge of NGC 6087 and is the brightest star in the region of the cluster (brighter than S Nor itself). Thus since it is a main sequence object it must lie in the foreground of the cluster and it is reassuring that the radial velocity result confirms this. In the case of 6087/t the radial velocity results do not give any very clear indication of membership. The star is classified as B9 III : it is peculiar in that it possesses hydrogen lines having very sharp absorption cores and moderately extensive wings.

The standard errors given for the BV–AV means in the two clusters (± 4 km/sec for M 25 and ± 3 km/sec for NGC 6087) include both observational errors and true velocity dispersion in the clusters. However since more than one spectrum of a good many of the stars is available it has been possible to estimate the standard errors of the means to be expected from observational errors alone. In both clusters this is found to be ± 3 km/sec so that the velocity dispersions are too small to be detected from the present results. In view of the broadness of the lines of the majority of main sequence cluster members, a very extensive programme would be necessary to obtain radial velocities of sufficient accuracy for studies of velocity dispersion.

Table II lists the spectral types in the two clusters. It also indicates the number of plates available with the two cameras. HD types are shown where these are available. In two cases, 6087/h and 6087/o, Hβ emission is suspected.

It is not profitable to enter into a discussion of the spectral types at this stage since magnitudes and colours of the stars will soon be published. However the spectral types alone do not indicate any abnormalities in the clusters. It is perhaps desirable to point out that, whilst the majority of stars are classified as late B–early A luminosity class V and have been referred to as main sequence stars, it is not beyond possibility that some of them may be luminosity class IV. Distinction between luminosity classes V and IV appears to be somewhat difficult in the case of broad-lined stars of early type.

TABLE II

Spectral Types in M 25

Irwin No.	C.P.D. No.	Spectra $f/2$	Spectra $f/3{\cdot}7$		HD	MK Type	
8	$-19°6875$	1	...	*		A1 V	
9	6876	2	...	*		B9 V	
10	6877	1	1	*		A0 V	
26	$-18°414$	...	1		K0	K2 V:	non-member
31	$-19°6879$	1	...	*		A1 V	
42	6884	1	...	*		B2 V:	
44	6889	1	...	*		B5 V:	
45	6886	2	...	*		B9 V	
49	6883	1	...			M1 III	member?
50	6881	...	1	*	B8	B6 V	
51	6882	...	1	*		A0 V:	
54	6890	2	...	*		B8 V	
67	6880	2	...	*		B7 V	
70	6888	...	1			A9 V	non-member
73	6885	1	...	*	A5	B6 V:	
80	6908	...	2	*	A0	B6 V	
91	6892	1	1	*	B9	B6 V	
93	6901	3	...	*		B8 V	
96	6895	2	...	*		B8 V	
97	6896	1	...	*		B7 V	
98	6897	2	...	*		B9 V	
99	6898	2	...	*		B8 V	
100	6902	2	...	*		B8 V	
101	6903	3	...	*		B8 V	
102	6904	3	1	*		B6 V	
109	6909	1	...	*		A1 V:	
116	6900	2	...	*		B8 V	
117	6899	1	...	*		B6 V	
120	6891	2	...	*		B8 V	
124	6893	1	...			A1 I	non-member?
128	6906	1	...	*		B8 V	
142	6905	1	...	*		A1 V	
149	6910	1	...	*		A0 V	
150	6915	1	2		K2	G5 III	
154	6925	1	...	*		A0 V:	
159	6912	1	...	*		B6 V	
163	6917	2	1	*	B9	B5 V	
168	6911	1	...	*		B5 V:	
190	6924	1	...	*		B8 V	
191	6922	1	...	*		A0 V	
192	6918	1	...			A0 I	non-member?
193	6921	...	2		K0	G2 III	

TABLE II (*continued*)

Spectral types in NGC 6087

Irwin No.	Cordoba No.	Spectra $f/2$	Spectra $f/3{\cdot}7$		HD	MK Type	
a	19	1	...	*		A1 V	
b	23	1	...	*	B9	B8 V	
c	27	1	...	*	A0	B9 V	
d	30	1	...	*	A	B8 V	
e	28	1	...		Ma	M3 III :	member?
f	26	3	1		K0	G4 III	non-member
g	33	1	...		A0	B8 V or III	member?
h	39	1	...	*		B8 V(e)	
i	40	1	1	*	B8	B5 V	
j	47	1	1	*	B5	B6 V	
k	61	1	...	*		B8 V	
l	45	1	...	*	B8	B5 V	
m	55	1	1	*		B5 V	
n	51	3	...	*		A0 V	
o	53	1	...	*	B8	B5 V(e)	
p	76	1	1	*	B5	B7 V	
q	78	1	...	*		A2 V:	
r	77	1	...	*		B8 V	
s	80	1	1	*	B8	B6 V	
t	84	1	1		B8	B9 III :	member?
u	83	1	...	*	A0	B9 V	
v	89	1	...		B9	A0 III : (V?)	member?
x	73	1	1		Ma	M1 III	
y	87	1	...	*		A0 V	
β	9	1	...		A2	A4 V	non-member

Notes to Table II:

(1) The stars used in forming the BV–AV means of Table I are denoted by an asterisk.

(2) The stars in M 25 are identified by their numbers in the Cape Photographic Durchmusterung (Cape Annals Vol. 3). The stars in NGC 6087 are identified by their numbers in the Cordoba Photographs of Clusters (Cordoba Resultados Vol. 19). NGC 6087 = Cordoba Δ326 (Norma).

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Radcliffe Observatory,
Pretoria:
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MOTIONS OF STARS OF TYPE A PERPENDICULAR TO THE GALACTIC PLANE

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Summary

Formal analogies are examined between density in directions perpendicular to the galactic plane and density in globular clusters. In the light of these, and otherwise, data on the density of A-stars as a function of z (including recent results published by van Rhijn) and the statistics of A-star radial velocities (from Wilson's recent catalogue) are examined, with the help of Jeans's theorem. It is concluded that the data demand a density of 0·2 solar masses per cubic parsec in the neighbourhood of the Sun.

1. The relations between density, potential and stellar velocities of the material in the neighbourhood of the Sun offer to a limited extent some analogies with the corresponding quantities in a globular cluster. Two methods of studying the former are open to us: we may either assume some morphology or model, and investigate the dynamical consequences of what we have assumed, or we may investigate the observed facts about stellar densities and velocities and find out as much as we can with the minimum appeal to theory. Since neither approach can as yet be entirely satisfactory, some sort of compromise may be attempted.

Considering briefly the first or model approach, we may make as a first approximation the very simple assumption that the material is stratified in parallel planes—planes, that is to say, parallel to the plane of symmetry of the Galaxy. Then the densities and motions perpendicular to this plane exhibit a marked resemblance to the densities and motions in a globular cluster. In fact if all the mass is collected into stars, or if in some way the residual matter (dust and gas) distributes itself as do the stars of mean density, we have simply to deal with a one-dimensional case of the globular cluster.

This may be quickly explored. Jeans's relation* is deduced from Liouville's theorem as follows. If

$$f(x, y, z, u, v, w)\, dx\, dy\, dz\, du\, dv\, dw$$

is the number of stars in the volume $dx\,dy\,dz$ having velocities between u and $u+du$ etc., then in a steady state, i.e. provided that f is not a function of the time,

$$Df/Dt = 0.$$

Jeans shows that if $\nu(V, \phi)\, dV$ is the number of stars per unit volume with speeds between V and $V+dV$ at a place where the gravitational potential is ϕ, the

* Jeans, J. H., *M.N.*, **76**, 78, 1915.

vectors being distributed isotropically, and if

$$\nu(V,\phi) \equiv V^2 f(V,\phi),$$

then

$$f(V,\phi) = F(V^2 - 2\phi) \qquad (1.1)$$

this being Jeans's relation.* Its equivalent may be deduced by enumerating stars in various z levels in the case where ϕ is a function of a single coordinate z alone (Oort) or in various concentric shells where ϕ is spherically symmetrical (Woolley), but the particular investigations make no more and no less appeal to "equilibrium" than does the deduction from Liouville's theorem: in all cases we simply assume that the state of the stellar motions is secularly stable. In the three-dimensional case we must also assume that the distribution of the vectors is isotropic, but if the analysis is restricted to a single coordinate this requirement vanishes.† However, if the velocities are Z restricted to the z coordinate, to obtain Jeans's relation we must set

$$\nu(Z,\phi) \equiv f(Z,\phi).$$

A particular solution of Jeans's relation is

$$F(Z^2 - 2\phi) = \text{const.} \times \exp[-j^2(Z^2 - 2\phi)],$$

where j is independent of Z and ϕ but may depend on the mass or class of the star. This is a requirement of full equilibrium, i.e. equilibrium stable against the effect of stellar collisions; but Oort employs a Gaussian distribution of stellar velocities (or a sum of such distributions) without supposing it to have been set up by stellar collisions. Instead, he infers it from radial velocity statistics.

Jeans's relation informs us that -2ϕ must enter the distribution in the same way as Z^2. If there are n classes of stars, of which the pth class has a Gaussian velocity distribution with a parameter j_p, then

$$F_p(Z^2 - 2\phi) = A_p \exp[-j_p^2(Z^2 - 2\phi)].$$

More generally, from a complete knowledge of the stellar velocity distribution we can construct the potential ϕ, or its derivative the acceleration $K(z)$.

To proceed to the analogies with globular clusters, we examine theoretically the supposition that ϕ is a function of z alone, in three cases.

Case I. Consider first the simple case where j has the same value for all stars. Let $z = 0$ be the plane of symmetry of the Galaxy and let ϕ_0 be the value of ϕ at $z = 0$. Write $2j^2(\phi - \phi_0) = -\psi$. Then

$$\nu(Z,\phi) = \text{const.} \times e^{-j^2Z^2} e^{-\psi}$$

and by integration with respect to Z, $\nu(\phi) = \text{const.} \times e^{-\psi}$. If ρ is the density and $\rho/\rho_0 = \eta$ then

$$\eta = e^{-\psi}.$$

We suppose that if there is any appreciable mass not concentrated into stars,

* So called by A. S. Eddington (*M.N.*, **76**, 572, 1916) a few months after Jeans published the original paper. But most writers quote the relation in a different form, for example W. M. Smart, *Stellar Dynamics*, pp. 324-326, 1938.

† The empirical facts concerning stars in the neighbourhood of the Sun are of course that the velocities are not distributed isotropically but ellipsoidally, and the observed ellipsoid does not coincide with the ellipsoid deduced from Jeans's relation for a rotating galaxy. (W. M. Smart, *Stellar Dynamics*, p. 395, equation (13); v. d. Pahlen, *Einführung in die Dynamik von Sternsystemen*, p. 126, footnote (18); S. Chandrasekhar, *Principles of Stellar Dynamics*, p. 125.) It is however quite natural to suppose that the state of the velocities in the galactic plane is not steady, yet the state of the velocities normal to this plane is steady, in the sense that the average numbers of stars crossing each plane parallel to the galactic plane are equal in the two senses upwards and downwards.

its density, as a function of z, is similar to that of the stars. Poisson's equation $\nabla^2\phi = -4\pi\Gamma\rho$ becomes

$$d^2\phi/dz^2 = -4\pi\Gamma\rho_0 e^{-\psi}$$

and if $l = (8\pi\Gamma\rho_0 j^2)^{-1/2}$ and $x = z/l$ we have

$$d^2\psi/dx^2 = e^{-\psi} \tag{1.2}$$

subject to the boundary conditions $\psi = 0$ and $\dot{\psi} = 0$ at $x = 0$. The solution is

$$\left.\begin{aligned} \psi &= 2 \ln \cosh (x/\sqrt{2}) \\ \eta &= e^{-\psi} = \operatorname{sech}^2 (x/\sqrt{2}) \end{aligned}\right\}. \tag{1.3}$$

Equation (1.3) corresponds to the "equation of the isothermal gas sphere" in the case of spherical symmetry* but the solution does not give an infinite mass in a cylinder of finite cross-section: we have in fact

$$\int_{-\infty}^{\infty} \rho \, dz = 2\sqrt{2} l \rho_0,$$

so that there is no need to invoke a "truncation" of the velocity distribution to get a finite mass.

From equation (1.3) we can calculate the periodic time of a star to move up and down in z. Let the motion be such that when furthest from $z = 0$ the potential is Ψ_0. At this point the velocity is zero and elsewhere

$$jZ = (\psi_0 - \psi)^{1/2}.$$

The periodic time T is therefore

$$T = 2\sqrt{2} \, . \, jl \int_0^{\psi_0} (\psi_0 - \psi)^{-1/2} (1 + e^{-\psi})^{-1/2} \, d\psi,$$

and since $2 > 1 + e^{-x} > 1$ for all positive x,

$$T = 4jl\psi_0^{1/2} \times c \tag{1.4}$$

where $\sqrt{2} > c > 1$, a formula which gives the order of magnitude of T without having to work out the integral.

Case II. We next consider the case where the stars have a distribution in mass and where j^2 is proportional to this mass. This is the full equilibrium case. Select a suitable mean mass $\overline{m}$ and let $\mu = m/\overline{m}$. Let ν_0 be the number of stars per unit volume at $z = 0$ and let the number of stars per unit volume with reduced masses between μ and $\mu + d\mu$ be

$$\nu_0 \xi_0(\mu) \, d\mu.$$

In particular we consider the case $\xi_0(\mu) = \text{const.} \times \mu^a e^{-b\mu}$. For each mass the velocity distribution is proportional to $\exp(-\mu \bar{j}^2 Z^2)$ *ex hypothesi*. Set

$$2\bar{j}^2(\phi - \phi_0) = -\psi.$$

Then Jeans's relation gives

$$\nu_\mu(Z, \psi) = \text{const.} \times e^{-\mu \bar{j}^2 Z^2} \, . \, e^{-\mu\psi}$$

and, by integration with respect to Z,

$$\nu_\mu(\psi) = \nu_0 \xi_0(\mu) e^{-\mu\psi}.$$

Since the density ρ is proportional to $\overline{m} \int_0^\infty \mu \xi_0(\mu) e^{-\mu\psi} \, d\mu$ we have

$$\rho = \text{const.} \times (b + \psi)^{-(a+2)} \int_0^\infty x^{a+1} e^{-x} \, dx$$

* *cf.* G. L. Camm, *M.N.*, **110**, 309, 1949.

and

$$\eta = \rho/\rho_0 = (1 + \psi/b)^{-(a+2)}. \tag{1.5}$$

Using, as before, $z = lx$ where now $l = (8\pi\Gamma\rho_0\bar{j}^{-2})^{-1/2}$, Poisson's equation gives

$$\frac{d^2\psi}{dx^2} = \left(\frac{b}{b+\psi}\right)^{a+2} \tag{1.6}$$

subject to $\psi = 0$ and $\dot{\psi} = 0$ at $x = 0$.

The value $a = 1$ fits the distribution of stellar mass in the neighbourhood of the Sun reasonably well, and we may set $b = 3$ which defines the mass $\bar{m}$. (Choice of $\bar{m}$ is a choice of l.)* Then (1.6) becomes

$$d^2\psi/dx^2 = (1 + \tfrac{1}{3}\psi)^{-3}$$

of which the appropriate solution is

$$\left.\begin{aligned} 1 + \tfrac{1}{3}\psi &= (1 + \tfrac{1}{3}x^2)^{1/2} \\ \eta = \rho/\rho_0 &= (1 + \tfrac{1}{3}x^2)^{-3/2} \end{aligned}\right\}. \tag{1.7}$$

The periodic time to move up and down is roughly $4\pi jl$ for moderate values of ψ_0.

Case III. Oort's empirical representation of the velocity statistics may be examined. Suppose there are n classes of stars and that for the pth class the z component of velocity is distributed as $\exp(-j_p{}^2Z^2)$. Suppose that the mass of each star in the pth class is m_p and that the abundance of the pth class at $z = 0$ is $\xi_0(p)$ subject to

$$\sum_{p=1}^{n} \xi_0(p) = 1.$$

By Jeans's theorem

$$\nu_p(Z, \phi) = \text{const.} \times \exp[-j_p{}^2(Z^2 - 2\phi)]$$

and

$$\begin{aligned} \eta = \rho/\rho_0 &= \sum_{p=1}^{n} m_p\xi_0(p)\exp[2j_p{}^2(\phi - \phi_0)] \Big/ \sum_{p=1}^{n} m_p\xi_0(p) \\ &= \sum_{p=1}^{n} A_p \exp[2j_p{}^2(\phi - \phi_0)] \end{aligned}$$

subject to $\sum_{p=1}^{n} A_p = 1$. Select a suitable value of $j_p{}^2$, namely $\bar{j}^2$, and set

$$a_p = j_p{}^2/\bar{j}^2, \qquad \psi = 2\bar{j}^2(\phi_0 - \phi), \qquad l = (8\pi\Gamma\rho_0\bar{j}^2)^{-1/2}, \qquad x = z/l.$$

Then Poisson's equation becomes

$$d^2\psi/dx^2 = \sum_{p=1}^{n} A_p \exp(-a_p\psi) \tag{1.8}$$

and we may impose the condition $\sum_{p=1}^{n} A_p a_p = 1$, which merely defines $\bar{j}^2$. In setting up this equation we suppose that ϕ is a function of z alone—that is, that the distribution of matter is uniform over planes normal to the z axis over distances large in comparison with the values of z with which we propose to deal. We

* In setting up a model. In a real analysis of stars we know too little about the mass function to be sure of $\bar{m}$.

also assume, as usual, that the dust and gas are distributed as are the stars of a given class (or a set of classes). The solution in series of (1.8) is

$$\psi = \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \dots$$
$$\eta = 1 - \tfrac{1}{2}x^2 + x^4 \left[\tfrac{1}{8} \sum_{1}^{n} A_p a_p{}^2 + \tfrac{1}{2} \cdot \tfrac{1}{4!}\right] - \dots . \tag{1.9}$$

The three solutions are very similar near $x=0$; we have in fact, from (1.3)

$$\psi = \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \dots , \qquad \eta = 1 - \tfrac{1}{2}x^2 + \tfrac{1}{6}x^4 - \dots ,$$

from (1.7)

$$\psi = \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \dots , \qquad \eta = 1 - \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 - \dots .$$

Numerical values of ψ and $d\psi/dx$ from equations (1.3) and (1.7) are given in Table I. It will be seen that they are very similar as far as the table goes: either set represents the potential and its derivative as well as can be required, in view of the doubtfulness, in any application to the Galaxy, of the legitimacy of leaving the dust out of account and of assuming uniformity in planes normal to the z axis. In Table 1, z = distance from galactic plane $= lx$; ϕ = gravitational potential, $d\psi = -2\bar{j}^2 d\phi$; $K(z)$ = acceleration $= -d\phi/dz = (2\bar{j}^2 l)^{-1}\, d\psi/dx$. In this $l = (8\pi\Gamma\rho_0\bar{j}^2)^{-1/2}$ where ρ_0 is the density at $z=0$ and $\bar{j}$ is the mean parameter in the velocity distribution.

TABLE I

Potential and acceleration for two solutions in which matter is distributed uniformly in infinite planes

Reduced distance from galatic plane of symmetry	Case I All velocity distributions alike. Equation (1·3)		Case II Equation (1·7)	
x	ψ	$d\psi/dx$	ψ	$d\psi/dx$
0	0·000	0·000	0·000	0·000
0·2	0·020	0·200	0·020	0·200
0·4	0·079	0·390	0·079	0·390
0·6	0·175	0·566	0·175	0·567
0·8	0·304	0·724	0·304	0·726
1·0	0·463	0·861	0·464	0·866
1·5	0·962	1·112	0·969	1·134
2·0	1·557	1·256	1·582	1·310
2·5	2·207	1·334	2·268	1·424
3·0	2·885	1·374	3·000	1·500
4·0	4·278	1·404	4·550	1·589
∞	∞	1·414	∞	1·732

The two solutions shown in Table I start together and diverge very little from each other even as far out as $x=2$. The acceleration $d\psi/dx$ derived from (1.7) is plotted against Oort's values* of $K'(z)$ in Fig. 1. Since Oort adopts $\log\rho = -az^2$, where $-a = 9\times10^{-6}$ when z is in parsecs, he implies a value of l. We have, from (1.7), for small x, $\ln\rho = -\tfrac{1}{2}x^2$. This is equivalent to Oort's value if $l=155$ parsecs, and this is adopted in drawing Fig. 1. A further constant must be chosen and in fact $d\psi/dx=1$ was taken to give $K'(z) = 3\times10^{-9}$ cm/sec^2. Then if $\bar{j}^{-1}$ is in km/sec and l is in cm, $(2\bar{j}^2 l)^{-1} = 3\times10^{-9}$ giving $\bar{j}^{-1} = 1{\cdot}71\times10^6$ cm/sec or 17·1 km/sec. The corresponding value of ρ_0 is 0·11 solar masses per cubic parsec, which is close to the value 0·09 given by Oort.

* Oort, J. H., *B.A.N.*, **6**, 262, 1932 (Table 14).

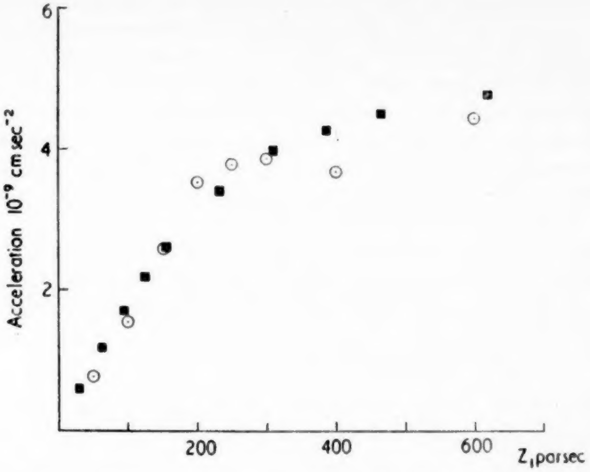

FIG. 1.—*Comparison between Oort's acceleration $K'(z)$ and $d\psi/dx$ from solution (7).*
⊙ *Oort's values.*
■ *Formula (7) with $l = 155$ pc and $(d\psi/dx)_\infty = 5{\cdot}2 \times 10^{-9}$ cm/sec^{-2}.*

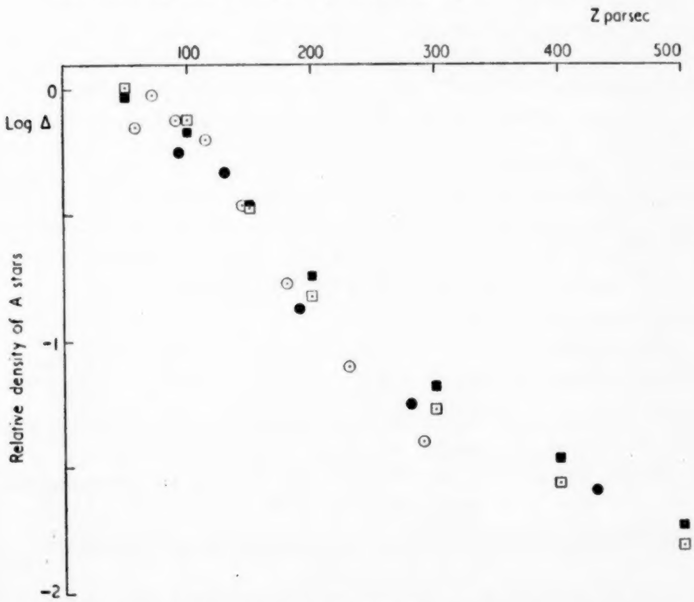

FIG. 2.—*Relative densities of A stars as a function of galactic height z.*
⊙ *Lindblad* ⊡ *Van Rhijn N.*
● *Counts in H.D.* ■ *Van Rhijn S.*

2. If however one makes use of the published data in a way differing from Oort's, one arrives at substantially different values of ρ_0. This is because the density that one finds is proportional to the square of a mean velocity that has to be adopted at some point.

Oort uses star counts by van Rhijn to find $\log\rho$ as a function of z for various absolute magnitude classes, for each of which he constructs a value of the mean square velocity as a function of z. This is a complicated matter, and one may avoid complication by considering instead of all stars (or all stars of a given absolute magnitude), all stars of a particular class that can be isolated in some way. At present the best data seem to be those given by A stars. For these the density as a function of height z has been given by Lindblad.* It may also be estimated from counts in the *Henry Draper Catalogue* (a device used by Oort) and it is the subject of a recent study by van Rhijn.† As the results agree quite well, one may have some confidence in the observed distribution. The results are shown in Fig. 2. The values derived from the *H.D.* counts require explanation. Oort selects the area from 11^h to $14^h 30^m$ in R.A. and from $+10^\circ$ to $+50^\circ$ Dec., with an extension to $+5^\circ$ Dec. between $11^h 30^m$ and 14^h, to represent the N galactic cap. Counts in this area in *H.D.* give

Visual magnitude	<5	5 to 5·99	6 to 6·99	7 to 7·99	8 to 8·99
Types A0 and A2	7	26	43	50	85
Types A3 and A5	3	13	31	38	58

These may be converted roughly to densities by using the simple assumption that all A0 and A2 stars have absolute magnitude $+0{\cdot}5$ and all A3 and A5 stars have absolute magnitude $+1{\cdot}5$. (Inspection of the count shows that this will work quite well, with approximately equal densities of stars in the two groups at each distance, since the number of A3 and A5 stars in the $(n+1)$th interval is always roughly equal to the number of A0 and A2 stars in the nth.) Then the A0 and A2 stars of apparent magnitude $5^m{\cdot}0$ are distant 79·4 parsecs and those of $6^m{\cdot}0$ are distant 126 parsecs, so that the stars between 5^m and 6^m lie in a volume proportional to 126^3 *minus* $79{\cdot}4^3$, etc. The mean density of stars of apparent magnitude $5^m{\cdot}5$, distant 100 pc from the Sun (and therefore 130 pc from the central plane of the Galaxy‡) is taken to be proportional to the number counted (26 in this case) divided by the relative volume $(126^3-79{\cdot}4^3)$ and so for other entries. In this way we find, combining the two groups,

Distance from the central plane in parsecs	93	130	190	280	430
log relative number of stars	$\bar{1}{\cdot}75$	$\bar{1}{\cdot}67$	$\bar{1}{\cdot}13$	$\bar{2}{\cdot}75$	$\bar{2}{\cdot}40$

The logarithms have an arbitrary added constant since the brightest stars are too few in number to give reliable densities for small values of z.

If the distribution of velocities of the A stars is a simple Gaussian one, say

$$n(Z)=\text{const.}\times e^{-\mu\bar{j}^2Z^2},$$

then the distribution of density with potential is, by Jeans's theorem,

$$\rho=\rho_0 e^{-\mu\psi}$$

where $\psi=2j^2(\phi_0-\phi)$, and then

$$\log\rho-\log\rho_0=-0{\cdot}4343\mu\psi,$$

so that the distribution of density with galactic height z in Fig. 2 gives a direct

* B. Lindblad, *Ark. Mat. Ast. Fys.*, **19b**, No. 15, 1927.

† P. J. van Rhijn, *Publ. Kapteyn Ast. Lab.*, Groningen, No. 57, 1955. Table 34.

‡ Throughout this paper it is assumed that the Sun is +30 pc from the central plane of the Galaxy, as used by J. H. Oort, *B.A.N.*, **236**, p. 260, 1932. P. J. van Rhijn finds a lower value.

determination of (a quantity proportional to) the potential ψ as a function of z. This may be compared with the potential in Table I, by adopting values of l and μ. In practice the quantity $l\mu^{-1/2}$ is fixed by the observations,* and $l = 150$ pc, $\mu = 2.5$ give a good fit to the observations from $z = 0$ to about $z = 250$ pc, at which point the potential deduced from the observed densities begins to diverge from the theoretical potential of Table I (equation (1.7)). This is shown in Fig. 3. Ignoring this divergence for the moment we have $l\mu^{-1/2} = 95$ parsecs. Since the average Z velocity is given by $|\bar{Z}| = (\pi\mu\bar{j^2})^{-1/2}$ and since $l^2 = (8\pi\Gamma\rho_0\bar{j^2})^{-1}$ we have

$$8\Gamma\rho_0 = \mu(|\bar{Z}|)^2 l^{-2}$$

so that with $l\mu^{-1/2} = 95$ pc we have the following table.

Mean velocity $\lvert\bar{Z}\rvert$ km/sec	Density ρ_0 Solar masses per cubic parsec
6	0·12
7	0·16
8	0·21
9	0·26

3. We now turn to the observed radial velocities of the A stars. Oort gives some results from data available when he wrote his paper which was published in 1932. There is now available an extensive catalogue† of radial velocities published by R. E. Wilson in 1953. This gives 124 stars of types A0 to A8 (including two which are double stars with one F-type component) in Oort's northern galactic cap area and 32 in the southern area. When these are corrected for the solar motion the velocities are grouped as shown in Table II.

TABLE II

Radial velocities of A stars in Oort's galactic cap regions corrected for the solar motion

Velocity km/sec	No. of stars N	S	Together		Gaussian distribution
0 to 4	40	10	50		54
5 to 9	51	9	60		48
10 to 14	17	9	26		31
15 to 19	6	1	7		15
20 to 25	8	1	9		6
30	...	1	1		$2\frac{1}{2}$
34	1	...	1		
36	1	...	1		
41	...	1	1		
Average velocity $\lvert\bar{Z}\rvert$, km/sec			7·9	9·6	8·3
Median velocity M km/sec			6·7	7·8	6·8

The Gaussian distribution shown is 156 stars with a median velocity of 6·8 km/sec, or $j = 0{\cdot}070$.

In forming Table II no correction was applied to allow for the fact that since the area extends to about 30° from the galactic pole, the radial velocities are not strictly speaking Z velocities. It is not possible to supply reliable Z components due to motion perpendicular to the line of sight, and it was simply supposed that the distribution is statistically similar to a Z distribution.

* because, for relevant values of ψ, $\psi \propto x^{1/2}$ approximately.

† R. E. Wilson, *General Catalogue of Radial Velocities*, Carnegie Institute of Washington *Publication* 601, 1953.

A survey was also made of radial velocities of stars brighter than $8^{m}\cdot0$, of types A0 and A0n only, within 40° of the galactic poles. The results (for velocities corrected for the solar motion are as follows.

Velocity km/sec	No. of A0 stars	Gaussian distribution
0 to 4	44	39
5 to 9	28	35
10 to 14	21	22
15 to 20	10	11
20 to 24	4	4
25 to 29	5	2
30 and above	2	$\frac{1}{2}$

Average velocity $|\bar{Z}|$ 8·9 km/sec.
Median velocity M 6·8 km/sec.

The Gaussian distribution has a median velocity of 6·8 km/sec.

These two surveys are not independent, as some of the data overlap, but the agreement is satisfactory. For a Gaussian distribution the average without regard to sign is 1·24 times the median, and $6\cdot8 \times 1\cdot24 = 8\cdot4$, in fair agreement for both surveys. Taking these results as they stand, the average velocity $|\bar{Z}|$ is between 8 and 9 km/sec, and from the table given at the end of Section 2, ρ_0 lies between 0·21 and 0·26 solar masses per cubic parsec—at least twice Oort's value. There is always the possibility of a selective effect in any catalogue of radial velocities, some faint stars having been put on a programme because of large proper motion: especially in the Southern Hemisphere where the catalogue is very incomplete. For this reason the median should be preferred to the mean, as a measure of the velocity distribution. However, the median is reduced slightly, but perhaps significantly, if one rejects velocities whose quality is given as c or d by Wilson, when the median velocity (of 54 A0 and A0n stars within 40° of the pole) drops to 6·0 km/sec.

4. In this section we discuss the effect of a double Gaussian velocity distribution, if present, on the density, and examine the evidence for a distribution of this kind. Since the work of Section 1 shows that the potential ψ as a function of reduced galactic height x is not sensitive to a choice of model velocity distribution we supposed that for $x \leqslant 2$ we may assume the values of ψ in Table II (Case II) as valid with sufficient accuracy. If now for a particular class of star

$$n(Z) = \text{const.} \times \{(1-\theta)j_1 e^{-j_1^2 Z^2} + \theta j_2 e^{-j_2^2 Z^2}\} \tag{4.1}$$

at $x=0$. Then by Jeans's theorem we must have elsewhere

$$n(Z) = \text{const.} \times \{(1-\theta)j_1 e^{-j_1^2 Z^2} e^{-\mu_1\psi} + \theta j_2 e^{-j_2^2 Z^2} e^{-\mu_2\psi}\} \tag{4.2}$$

where $j_1^2 = \mu_1 j^2$ and $j_2^2 = \mu_2 j^2$. Integrating with respect to the velocity Z we have, on division,

$$\eta = \rho/\rho_0 = e^{-\mu_1\psi}\{1 + \theta(e^{(\mu_1-\mu_2)\psi} - 1)\} \tag{4.3}$$

the relative density η referring to the particular class of star.

Oort assumes a distribution of A-type velocities in which $\theta = 0\cdot043$ and $j_2/j_1 = 20/84$. For such a case, equation (4.3) gives values of η which are very similar to that given by a single Gaussian distribution, namely

$$\eta = e^{-\mu\psi}$$

for small values of ψ, but shows important differences when ψ becomes larger. The following short table shows how with a suitable adjustment of the constants the partial densities can be made practically identical out to $z=200$ parsecs.

TABLE III

Star densities arising from single and double Gaussian velocity distributions

		Single Gaussian distribution		Double Gaussian distribution	
x	ψ	z, pc	$-\log \eta$	z, pc	$-\log \eta$
0·8	0·304	120	0·33	132	0·31
1·0	0·464	150	0·50	140	0·47
1·5	0·969	225	1·05	210	0·91
2·0	1·582	300	1·72	280	1·28
2·5	2·268	375	2·46	350	1·46
3·0	3·000	450	3·26	420	1·52

In this we have taken $l=150$ pc and $\mu=2{\cdot}5$ for the single distribution, and $l_1=140$ parsecs and $\mu_1=2{\cdot}5$ for the double distribution. If the velocity distributions give the same average velocity at $z=0$, then

$$j_1=1{\cdot}138j$$

and the central density (of the whole material) deduced from space and velocity distributions is reduced by about ten per cent when passing from the single distribution to the double distribution (i.e. in the ratio $j^2l^2/j_1{}^2l_1{}^2$). Fig. 3, in which both theoretical distributions are plotted against the observed densities of A stars, shows that the double velocity distribution gives a much better representation than does the single distribution, for $z>200$ parsecs.

But there is no real support for the double Gaussian distribution in the statistics of A star radial velocities: in fact the contrary, since from equation (4.2) we can easily show that the average velocity should increase quite markedly with increasing ψ. Since this corresponds to increasing z or increasing distance from the Sun, the double Gaussian distribution requires an increase of average velocity with increasing stellar magnitude, and this is not observed. We have in fact

$$|\bar{Z}|=\text{const.}\times\{1+0{\cdot}193e^{(\mu_1-\mu_2)\psi}\}/\{0{\cdot}957+0{\cdot}043e^{(\mu_1-\mu_2)\psi}\}.$$

Now assume that all A stars have absolute magnitude $+1$: then the distance of those of apparent magnitude $6^{\mathrm{m}}{\cdot}0$ is 100 parsecs, or $z=130$ pc and we have:

Apparent magnitude	z, parsecs	$\mu_1\psi$	Average velocity km/sec
$6^{\mathrm{m}}{\cdot}0$	130	0·50	8·8
$7^{\mathrm{m}}{\cdot}0$	188	1·82	10·2
$7^{\mathrm{m}}{\cdot}5$	230	2·34	15·2

But the average velocity of 44 stars with m (vis)$<5^{\mathrm{m}}{\cdot}9$ is $|\bar{v}|=7{\cdot}8$ km/sec, and the average velocity of 25 stars with m (vis)$>7^{\mathrm{m}}{\cdot}0$ is $|\bar{v}|=8{\cdot}0$ km/sec, and these figures are very strongly against the increase of velocity with distance required by the double Gaussian distribution.

5. If we reject the double Gaussian velocity distribution and accept the single Gaussian distribution as a satisfactory representation of the velocities, then Fig. 2 defines the behaviour of ψ and it is quite clear that ψ as a function of z has a point of inflection. This is impossible if the material is distributed uniformly in parallel planes, whether it consists of stars which can be dealt with by the methods of Section 1 or whether it consists partly of dust and gas. But of course there is nothing to say that the material of the Galaxy is distributed uniformly in parallel planes—quite the reverse: there are spiral arms. This suggests that we examine the potential of matter distributed in (infinitely) long cylinders. We shall have to abandon Poisson's equation and consider mere models, because the spiral arms are certainly not equilibrium figures unless galactic rotation is taken into account, and not even then, considering the difficulty of accounting (on equilibrium theory) for the observed axes of the Schwarzschild velocity ellipsoid. In other words we abandon the theoretical approach of Section 1 and seek a compromise between some vestige of theory (Jeans's theorem) and the empirical facts. This consists of setting up as a model "spiral arm" an infinitely long cylinder of square section with a uniform density.

The potential of an infinitely long cylinder of uniform density ρ and square section of side $2a$ at a point distant z from the centre of the square, situated along a line passing through the centre normal to a face, is given by

$$\left.\begin{aligned}\phi_0-\phi &= 4\Gamma\rho a^2[F(z/a+1)-F(z/a-1)-2F(1)] \quad \text{for } z>a\\ &= 4\Gamma\rho a^2[F(1+z/a)+F(1-z/a)-2F(1)] \quad \text{for } z<a\end{aligned}\right\} \tag{5.1}$$

where ϕ_0 is the potential at the centre and

$$F(x) = \tfrac{1}{2}(x^2-1)\cot^{-1}x+\pi/4-\tfrac{1}{2}x[1-\ln(1+x^2)].$$

A point of inflection in ϕ occurs where $z=a$, and by inspection of Fig. 2 if this model is to represent the observed potential we must set $a=150$ parsecs (or possibly a little larger). Since $F(2)=2{\cdot}090$, $F(1)=0{\cdot}632$ and $F(0)=0$, we have, when $z=a$,

$$\phi-\phi_0=4\Gamma\rho a^2\times 0{\cdot}826.$$

If as usual we set $\psi=2j^2(\phi_0-\phi)$, we have from inspection of Fig. 2 $\log\rho=-0{\cdot}5$ at $z=150$ pc or $\mu\psi=1{\cdot}1513$ at $z/a=1$. Then

$$\left.\begin{aligned}\mu\psi &= 1{\cdot}394[F(z/a+1)-F(z/a-1)-2F(1)] \quad z>a\\ &= 1{\cdot}394[F(1+z/a)+F(1-z/a)-2F(1)] \quad z<a\end{aligned}\right\}. \tag{5.2}$$

Points from equation (5.2) are plotted as crosses in Fig. 3. The model represents the potential deduced from A-star densities extremely well, and appears to indicate that the effective width of the spiral arm in which the Sun is embedded is of order $2a$ or 300 parsecs. Since

$$\mu\psi=2\mu j^2(\phi_0-\phi)=8\mu j^2\Gamma\rho a^2\times 0{\cdot}826 \quad \text{at} \quad z=a,$$

we have

$$\rho=40{\cdot}5(a^2\mu j^2)^{-1}=40{\cdot}5(a^2j_\mu{}^2)^{-1}$$

if j is in $(\text{km/sec})^{-1}$ and a in parsecs. This gives the following table:

median velocity M km/sec	j_μ $(\text{km/sec})^{-1}$	ρ, solar masses per cubic parsec
6·8	0·070	0·37
6·0	0·080	0·28
5·3	0·090	0·22

the densities being higher than before; since the attracting figure is now limited in extent, it requires a greater density to produce a given potential than does the unlimited figure.

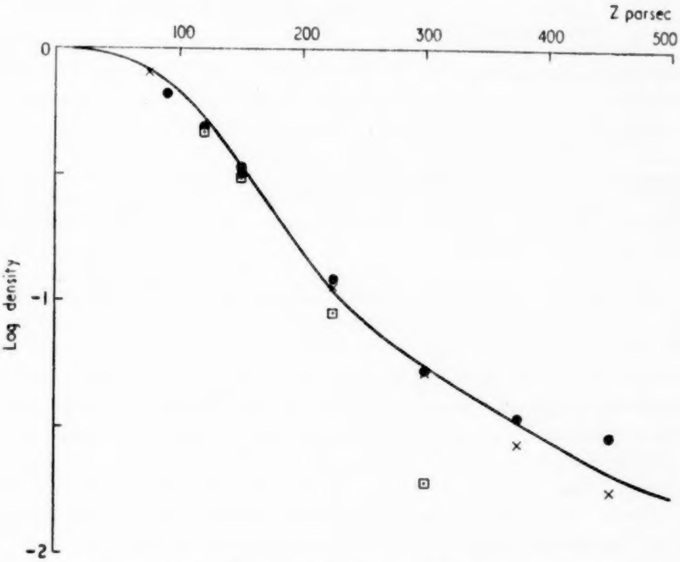

FIG. 3.—*Theoretical distributions of A stars with z.*

● *Double Gauss* ⊡ *Single Gauss*
× *Square cylinder* — *Curve representing observations shown in Fig. 2.*

6. The distribution of radial velocities as actually observed differs from the true distribution on account of observational errors. In this section we make an attempt to determine an appropriate correction so as to infer the true distribution from the observed distribution.

If a number of stars are members of an open cluster they may be supposed to have a common velocity with a very small dispersion about it. For example, the dispersion in velocity in the stars in the Pleiades is less than 1 km/sec.* Nevertheless the radial velocities of the A stars in the Pleiades in Wilson's catalogue, differenced from the mean radial velocity of such stars, show a median of 2·6 km/sec and a root mean square of 3·0 km/sec: that these differences are almost entirely due to observational error and not due to a true velocity dispersion in the cluster is shown by the proper motions.

Besides the Pleiades, there are three other clusters (Hyades, Coma Berenices and Praesepe) which give useful data about the dispersion of the catalogued A star radial velocities. Wilson assigns four qualities to the determinations, a, b, c and d, in descending merit, eighty per cent of the stars being classed either b or c: but the clusters do not show a difference of quality between the b and c class velocities.

* R. v. d. R. Woolley, *M.N.*, **116**, 296, 1956, quoting spectroscopic material given by Struve and proper motions given by Hertzsprung and by Titus.

Collected results are as shown in Table IV.

TABLE IV

Dispersion in catalogued radial velocities of A stars in open clusters

Cluster	Pleiades	Praesepe	Coma	Hyades
No. of A stars in Wilson's catalogue	35	18	11	27
median velocity difference, km/sec	2·6	2·8	2·0	1·9
root mean square, km/sec velocity difference	3·0	3·9	3·0	3·3

The median refers to the difference, without regard to sign, between the observed velocity of the star and the mean velocity of the A stars in the cluster, except in the case of the Hyades, where the observed velocity is differenced from the calculated $v \cos \lambda$ velocity according to van Bueren.

The evidence for a possible difference between the dispersions in class b and class c velocities is inconclusive, but as far as it goes it does not support any such difference. Only four stars in the Hyades and one in Coma are of quality c (and none in either case is of quality d). In Praesepe the c class velocities show *less* dispersion than do the b class velocities. In the Pleiades there are 13, 15 and 7 velocities of classes b, c, and d, with practically identical dispersions. In all cases comparatively few stars of classes A0 and A1 occur. In the Hyades and Praesepe the A stars are late in type. In the Pleiades the dispersion increases from early A to late A type: the median is 2·0 km/sec for 23 stars of types A0, A1 and A2, compared with 2·6 km/sec for 35 stars of all A types.

From these results we conclude that a root mean square dispersion of ± 3 km/sec on account of observational error can be ascribed to the catalogued A star velocities. With this figure we compute the following corrections:

Observed median velocity, km/sec	6	6·5	7
Observed value of j, (km/sec)$^{-1}$	0·0795	0·0734	0·0681
Corrected value of j, (km/sec)$^{-1}$	0·0748	0·0698	0·0692
Correction to the deduced density: multiply by	0·89	0·90	0·92

The deduced density is accordingly reduced by about ten per cent, by allowing for the observational dispersion in radial velocities, taken to be ± 3 km/sec. (The reduction would be approximately doubled if the dispersion were taken as ± 4 km/sec.)

We now summarize the data concerning the median velocity, as follows:

156 A stars of all types in Oort's galactic cap regions	6·7 km/sec.
115 A0 stars within 40° of the galactic poles	7·2 km/sec.
55 A0 stars within 40° of the galactic poles a+b quality only	6·6 km/sec.

If the adopted median velocity is taken as 6·7 km/sec, and it is assumed that the observational dispersion has a r.m.s. value of 3·0 km/sec, then

$$j = 0{\cdot}0747 \text{ (km/sec)}^{-1}$$

and with this value of j we find

$$\rho_0 = 0{\cdot}18 \text{ solar masses per cubic parsec.}$$

This assumes that the velocity distribution is a single Gaussian distribution and that the material is stratified in parallel planes.

This density is high in comparison with values given by J. H. Oort* which range from 0·11 to 0·08 solar masses per cubic parsec, though Lindblad† gave 0·22 in these units and Jeans‡ 0·14. The value given here might be reduced if evidence could be adduced for the reality of a double Gaussian distribution in A star velocities, but would have to be increased if any allowance was necessary for departure from stratification of galactic material in parallel planes: and this departure must occur. The writer is inclined to the view that if evidence from stellar classes other than type A forces us to adopt a density smaller than, say, 0·15 solar masses per cubic parsec it will be necessary to suppose that the A stars do not satisfy Jeans's relation. This would be the case, for example, if they were an expanding system, but it does not seem to be an hypothesis that one would lightly adopt.

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* J. H. Oort, *B.A.N.*, **6**, No. 238, 284, 1932.
† B. Lindblad, *Upsala Medd.*, No. 11, p. 30, 1926.
‡ J. H. Jeans, *M.N.*, **82**, 122, 1922.

GENERAL RELATIVITY AND MACH'S PRINCIPLE

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Summary

Sciama, among others, has taken the view that general relativity has failed to account satisfactorily for the inertial properties of matter. This paper shows that general relativity is entirely consistent in principle with Sciama's ideas of inertia as an inductive effect predominantly of distant matter, and that therefore his remarks concerning general relativity are not justified. It is shown that general relativity provides a superior presentation of his idea of Mach's principle and appears to be the general tensor theory he was looking for.

Arguments are put forward to show that general relativity may fully incorporate Mach's principle contrary to Einstein's own belief. This paper emphasizes the fitness of the steady state theory as a cosmological solution which permits this possibility.

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1. *Introduction.*—A tentative theory has been presented by Sciama (1), with Maxwell type equations, which is designed to provide a combination of Newton's laws of motion and of gravitation with the inertial frames determined by Mach's principle. In the introduction to his paper Sciama states that general relativity has failed to provide an adequate theory of inertia. He claims that his theory differs from general relativity principally in the following respects:

(i) it enables the amount of matter in the universe to be estimated from a knowledge of the gravitational constant;

(ii) the principle of equivalence is a consequence of his theory, not an initial axiom; and

(iii) it implies that gravitation must be attractive.

The chief characteristic of Sciama's theory is that "in the rest frame of any body the gravitational field of the universe as a whole cancels the gravitational field of local matter so that in this frame the body is 'free'. Thus in this theory inertial effects arise from the gravitational field of a moving universe." For this purpose Sciama employs a scalar potential Φ and a vector potential $\mathbf{A}$ to calculate gravitational effects, using Maxwell type field equations in flat space-time.

It is the purpose of this paper to show that general relativity is fully consistent with this interpretation of Mach's principle by Sciama, and to indicate that general relativity may fully incorporate Mach's principle.

2. *Free particle in general relativity.*—The motion of a free particle in general relativity, when the gravitational field is weak and when the reference frame is such that the spatial velocity of the particle is small compared with the velocity of light, can be described by a Maxwell-type pondermotive equation. This idea

is not new and has in fact, with limited application, been presented by Einstein (2). But since Einstein's derivation of the result appears to contain errors of detail we give our own derivation here, before investigating its significance for Mach's principle.

Let Latin letters indicate space coordinates running over indices 1, 2, 3 while Greek letters cover the space-time indices 1, 2, 3, 4. The world line of a free particle in general relativity is a geodesic in the field having equations, in a standard form,

$$\frac{d}{ds}\left(g_{\mu\alpha}\frac{dx^\alpha}{ds}\right)=\tfrac{1}{2}\frac{\partial g_{\alpha\beta}}{\partial x^\mu}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds} \tag{1}$$

where

$$ds^2=g_{44}(dx^4)^2+2g_{4p}dx^4dx^p+g_{pq}dx^pdx^q. \tag{2}$$

For $\mu=i$ equations (1) may be written

$$\left(\frac{ds}{dx^4}\right)^2\frac{d}{ds}\left(g_{ip}\frac{dx^p}{ds}\right)=\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i}-\left(\frac{ds}{dx^4}\right)^2\frac{d}{ds}\left(g_{i4}\frac{dx^4}{ds}\right)+\frac{\partial g_{p4}}{\partial x^i}\frac{dx^p}{dx^4}+\tfrac{1}{2}\frac{\partial g_{pq}}{\partial x^i}\frac{dx^p}{dx^4}\frac{dx^q}{dx^4}.$$

Write now $x^4=t$, $dx^p/dt=v^p$, and neglect squares and products of the spatial coordinate velocities v^p, getting

$$\frac{ds}{dt}\frac{d}{dt}\left(g_{ip}\frac{dt}{ds}v^p\right)=\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i}-\frac{ds}{dt}\frac{d}{dt}\left(g_{i4}\frac{dt}{ds}\right)+\frac{\partial g_{p4}}{\partial x^i}v^p. \tag{3}$$

We shall now represent the metric (2) as that of a weak field in the form

$$\begin{aligned}ds^2=(1+\gamma_{44})dt^2+2\gamma_{4p}dx^pdt-(1-\gamma_{11})(dx^1)^2-(1-\gamma_{22})(dx^2)^2-(1-\gamma_{33})(dx^3)^2\\+\gamma_{pq}dx^pdx^q \qquad (p\neq q)\end{aligned} \tag{4}$$

Here we take the velocity of light, c, as unity. We see that the $\gamma_{\mu\nu}$ are the deviations of the $g_{\mu\nu}$ from the Galilean values in the so-called inertial frames. They are the $\gamma_{\mu\nu}$ of Einstein's treatment of the problem except for sign due to his employment of imaginary x^4.

We make the assumption that the squares and products of the $\gamma_{\mu\nu}$ and those of their derivatives can be neglected. In solving the field equations to this approximation Einstein showed (3) that the $\gamma_{\mu\nu}$ were the solutions of the equations

$$\left\{\left(\frac{\partial}{\partial x^1}\right)^2+\left(\frac{\partial}{\partial x^2}\right)^2+\left(\frac{\partial}{\partial x^3}\right)^2-\left(\frac{\partial}{\partial t}\right)^2\right\}\gamma_{\mu\nu}=2\kappa(T_{\mu\nu}-\tfrac{1}{2}\delta_{\mu\nu}T) \tag{5}$$

provided

$$\frac{\partial^2}{\partial x^\nu\partial x^\alpha}(\gamma^\alpha_\mu-\tfrac{1}{2}\delta^\alpha_\mu\gamma^\beta_\beta)+\frac{\partial^2}{\partial x^\mu\partial x^\alpha}(\gamma^\alpha_\nu-\tfrac{1}{2}\delta^\alpha_\nu\gamma^\beta_\beta)=0 \tag{6}$$

to the order of the approximation. Here $\kappa=8\pi G/c^2$ where G is the Newtonian constant of gravitation, and $\gamma^\alpha_\mu=\delta^{\alpha\beta}\gamma_{\mu\beta}$ where $\delta^{\alpha\beta}$ are the Galilean values of the $g^{\alpha\beta}$. Assuming the contribution of stress to the energy momentum tensor to be vanishingly small compared with the density and momentum components for the case considered by Einstein, equation (5) yields Einstein's solution:

$$\left.\begin{aligned}&\gamma_{11}=\gamma_{22}=\gamma_{33}=\gamma_{44}=-\frac{\kappa}{4\pi}\int\frac{[\rho]}{r}dV,\quad \gamma_{4p}=\frac{\kappa}{2\pi}\int\frac{[\rho u^p]}{r}dV\\ \text{and}\\ &\gamma_{pq}=0,\qquad\qquad p\neq q\end{aligned}\right\}. \tag{7}$$

In this solution ρ is the mass density, u^p the space velocity, of the element of mass in volume dV at distance r from the point where the $\gamma_{\mu\nu}$ are evaluated, all quantities being measured by observers at rest in the reference frame. Square

brackets indicate retarded values corresponding to the propagation of the field with the unit velocity.

The integrals, supposed convergent, are over all matter producing the field. Such a solution of the wave equation of Lorentz (equation (5)) is well known to be valid only if the quantities solved for (the $\gamma_{\mu\nu}$) tend to zero in a suitable way. Thus it is clear that the case considered by Einstein involves mass concentrations only in the neighbourhood of the space origin and a field metric which is Galilean at "infinity". The integrals in (7) evaluated over such mass concentrations are therefore evidently convergent. The solution has to be consistent with conditions (6) which will be satisfied if the expressions

$$\frac{\partial}{\partial x^\alpha}(\gamma^\alpha_\mu - \tfrac{1}{2}\delta^\alpha_\mu \gamma^\beta_\beta)$$

vanish, for all μ, to the first order in the $\gamma_{\mu\nu}$. Using (7) and the fundamental equations $T,^{\mu\nu}_{\nu}=0$, it is easily seen that for integrals over a finite region of mass these expressions are indeed second order quantities.

To this approximation therefore, retaining only first order terms, we can reduce equation (3) to

$$\frac{d}{dt}\left(\frac{g_{ii}v^i}{\sqrt{g_{44}}}\right) = \tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i} - \frac{d}{dt}(g_{i4}) + \frac{\partial g_{p4}}{\partial x^i}v^p,$$

on using (2) to find the appropriate approximation for ds/dt in each term. There is of course no summation over i on the left. Rearranging we can write:

$$\frac{d}{dt}\left(-\frac{g_{ii}v^i}{\sqrt{g_{44}}}\right) = -\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i} - \frac{\partial}{\partial t}(-g_{4i}) + \left\{\frac{\partial}{\partial x^p}(g_{4i}) - \frac{\partial}{\partial x^i}(g_{4p})\right\}v^p \tag{8}$$

so that

$$\frac{d}{dt}\{(1-\gamma_{ii}-\tfrac{1}{2}\gamma_{44})v^i\} = -\tfrac{1}{2}\frac{\partial \gamma_{44}}{\partial x^i} - \frac{\partial}{\partial t}(-\gamma_{4i}) + \left\{\frac{\partial}{\partial x^p}(\gamma_{4i}) - \frac{\partial}{\partial x^i}(\gamma_{4p})\right\}v^p. \tag{9}$$

Write now

$$\phi = -G\int\frac{[\rho]dV}{r},\quad A^p = -4G\int\frac{[\rho u^p]}{r}dV \tag{10}$$

so that by (7),

$$\gamma_{11}=\gamma_{22}=\gamma_{33}=\gamma_{44}=2\phi,\quad \gamma_{4p}=-A^p. \tag{11}$$

Equation (9) can therefore be written in vector form covering $i=1, 2, 3$

$$\frac{d}{dt}\{(1-3\phi)\mathbf{v}\} = -\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\wedge \operatorname{curl}\mathbf{A}. \tag{12}$$

Equation (12) is the required Maxwell-type pondermotive equation of the field. The assumptions made during its derivation are:

(i) The particle velocity $\mathbf{v}$ in the reference frame is assumed small such that v^2/c^2 is negligible compared with v/c.

(ii) The deviations of the $g_{\mu\nu}$ from the Galilean values are small such that their squares and products and those of their derivatives can be neglected.

(iii) The deviations $\gamma_{\mu\nu}$ vanish at "infinity" so that the quantities $\mathbf{A}$, ϕ are defined in terms of convergent integrals. If *in addition* we now further assume that

(iv) The source velocities of the field are also small in the reference frame so that the same remark as in (i) applies for them, then equation (12) reduces to

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v}\wedge\operatorname{curl}\mathbf{A}, \tag{13}$$

The equation obtained by Einstein was (our notation)

$$\frac{d}{dt}[(1-\phi)\mathbf{v}] = -\operatorname{grad}\phi - \frac{\partial \mathbf{A}}{\partial t} + \mathbf{v} \wedge \operatorname{curl} \mathbf{A}.$$

Since he assumed condition (iv) as well as (i), (ii), (iii), his result is incorrect to the order he was considering, and misleading. In obtaining this result he put

$$[p4, i] = \tfrac{1}{2}\left(\frac{\partial g_{4i}}{\partial x^p} - \frac{\partial g_{4p}}{\partial x^i}\right)$$

thereby neglecting the term $\partial g_{ip}/\partial x^4$ which, when $p=i$, contributes to our result in equation (9) as the term $d(-\gamma_{ii})/dt$ in the coefficient of v^i in the left hand side. The neglect of this term is of course consistent with condition (iv), but on the other hand the retention of the term $d(-\phi)/dt$ in the coefficient of $\mathbf{v}$, arising in our approximation from the term $d\,(-\tfrac{1}{2}\gamma_{44})/dt$ in the coefficient, is not consistent with Einstein's assumptions.

3. *Interpretation of the pondermotive equation.*—As Einstein pointed out, equation (12) indicates that general relativity goes far towards incorporating Mach's principle. It may be compared with the Newtonian equation, viz.,

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad}\phi.$$

The additional terms are small in the quasi-Galilean frame considered by Einstein, and, as he said, beyond physical measurement. Nevertheless they show in the sense of Mach's principle how concentrated matter affects the inertial mass of a freely moving particle, and the acceleration of its locally inertial rest frame relative to the given frame, in the following respects:

(i) The inertial mass is apparently proportional to $1-3\phi$.

(ii) The locally inertial rest frame of the particle is accelerated by means of:

(*a*) gravitational attraction towards the local mass concentrations indicated by the term $-\operatorname{grad}\phi$;

(*b*) an inductive effect of local accelerating matter in the same sense as the acceleration, indicated by the term $-\partial\mathbf{A}/\partial t$;

(*c*) an inductive effect of matter which is rotating relative to the compass of inertia (to use Gödel's phrase) at "infinity", in the sense of the rotation, as indicated by the term $\mathbf{v} \wedge \operatorname{curl}\mathbf{A}$. This is of the same type as the "fictitious" Coriolis force familiar in Newtonian dynamics, when a reference frame is used which is rotating relative to the compass of inertia. Centrifugal force also arises in this case as a fictitious gravitational force.

It is clear therefore that general relativity certainly incorporates in detailed manner the aspects of Mach's principle indicated above. For a satisfactory theory of Mach's principle, however, Einstein realised the necessity of showing how inertia depended on the entire cosmic distribution of matter. Because he assumed that the metric was Galilean at "infinity" and therefore excluded any contribution to $\mathbf{A}$ and ϕ other than that of the matter concentrated near the space origin, he was unable to examine the cosmic influence on inertia.

We shall here put forward an analysis to show that general relativity actually permits the same interpretation of inertia which has been presented by Sciama as the inductive effect of the whole universe.

4. *Inductive effect of the universe in general relativity.*—We shall investigate the extent to which we may generalize the circumstances when the motion of a free particle may be described by a Maxwell-type pondermotive equation. For this purpose we make the assumptions less restrictive than in the quasi-Galilean case as follows:

(i) The particle velocity $\mathbf{v}$ in the reference frame is assumed small such that v^2/c^2 is negligible compared with v/c.

(ii) The velocities of the sources of the field, in the region of space-time coordinates with which we shall be concerned, are also small of the same order so that the same remark applies.

(iii) The deviations of the $g_{\mu\nu}$ from the Galilean values are small in the above quoted range of space-time coordinates, such that their squares and products and those of their derivatives can be neglected. We do not however assume that these deviations vanish at "infinity", nor that they even remain small outside the specified range.

It is clear from equation (8) that the equation of motion of a free particle can in these circumstances be written

$$\frac{dv^i}{dt} = -\tfrac{1}{2}\frac{\partial g_{44}}{\partial x^i} - \frac{\partial}{\partial t}(-g_{4i}) + \left\{\frac{\partial}{\partial x^p}(g_{4i}) - \frac{\partial}{\partial x^i}(g_{4p})\right\}v^p \tag{14}$$

for $i = 1, 2, 3$.

This equation is generally covariant, in the sense that, in all reference frames and regions of space-time which do not violate the assumptions above, it describes the space motion of the free particle in terms of the derivatives of the $g_{\mu\nu}$ involved. We now generalize the quantities $\mathbf{A}$, ϕ occurring in the quasi-Galilean analysis by defining

$$(\mathbf{A}, \Phi) \equiv (-g_{4i}, \tfrac{1}{2}g_{44}). \tag{15}$$

The three equations in (14) may then be written concisely

$$\frac{d\mathbf{v}}{dt} = -\operatorname{grad}\Phi - \frac{\partial\mathbf{A}}{\partial t} + \mathbf{v}\wedge\operatorname{curl}\mathbf{A}. \tag{16}$$

The vector notation implies the vector character of the terms for purely spatial transformations. For space-time transformations however the quantities $(\mathbf{A}, \Phi)$ do not transform as a 4-vector but as components of the tensor $g_{\mu\nu}$. This is because, unlike the corresponding electromagnetic pondermotive equation, the permitted transformations are not necessarily between inertial frames and therefore not in general linear.

It is to be noted that here we have not as in the quasi-Galilean case identified $\mathbf{A}$, Φ with the deviations of the $g_{\mu\nu}$ involved, from their Galilean values but, consistent with our endeavour to account for the whole of inertia according to Mach's principle, in terms of the total $g_{\mu\nu}$. The covariance of (16) is secured by the tensor character of the total $g_{\mu\nu}$ involved; the deviations do not transform as tensors for general transformations. Indeed according to the field equations it is the total $g_{\mu\nu}$ field that is related inseparably to the distribution of matter in the whole universe.

Bearing in mind therefore the physical interpretation of the quantities $\mathbf{A}$, ϕ in the quasi-Galilean case we should expect analogous interpretation of $\mathbf{A}$, Φ in (16) which would, if Mach's principle is to be satisfied, take account of the distribution and motion of matter in the whole universe, relative to the particular

reference frame being used. It would be an immediate consequence of such an interpretation of the terms in (16) that the "fictitious" forces of Newtonian mechanics in accelerating or rotating reference frames would become directly attributable to the inductive effect of a moving universe.

In particular, in a reference frame in which a freely moving particle was permanently at rest, equation (16) would reduce to

$$-\operatorname{grad}\Phi - \frac{\partial \mathbf{A}}{\partial t} = 0, \tag{17}$$

holding at the particle. This is the equation *postulated* by Sciama. To use Sciama's expression the "gravoelectric" field of the whole universe would be zero at the particle and it would be gravitationally "free" in its own rest frame.

For a reference frame at rest relative to the averaged motions of the rest of the matter in the universe (the "smoothed-out" universe) we should expect by Mach's principle that, in the neighbourhood of the space origin, the derivatives of $\mathbf{A}$, Φ on the right of (16) would vanish and therefore that the left hand side must vanish. The real existence of such frames which are locally inertial is the basis of Newtonian mechanics. This aspect of Mach's principle is built into general relativity theory since the field equations predict that such a reference frame will be Galilean near the space origin, because of the spherical symmetry about it. Thus in this neighbourhood the metric will approximate to

$$ds^2 = dt^2 - dx^2 - dy^2 - dz^2. \tag{18}$$

It is emphasized that in this paper we attach importance for Mach's principle to the total $g_{\mu\nu}$ involved in equation (16) and not just their derivatives. General reasons for this have already been given and further justification provided in Sections 5, 6. Accordingly it is important to obtain the total value of Φ. It follows from equation (18) that the static potential Φ_0, at the origin of such a frame, of the whole universe would be

$$\Phi_0 = \tfrac{1}{2} g_{44}(0) = \tfrac{1}{2} \text{ (or } \tfrac{1}{2}c^2 \text{ in general units).} \tag{19}$$

The dimensions of Φ and the significance we are trying to associate with it would require Φ_0 to be of order $-GM/R$ where M is the effective gravitational mass of the universe and R its effective radius. Sciama's approach is to *define* Φ_0 as $-\int_{r=0}^{r=R} \frac{\sigma dV}{r}$ where σ is the gravitational mass density, and he gets

$$G\Phi_0 = -c^2.$$

Both results are numerically of the same order. The discrepancy in sign will occupy us later. Before investigating to what extent general relativity theory justifies this tentative physical interpretation of $\mathbf{A}$, Φ, we give some applications of our theory.

5. *Applications of the inductive theory in general relativity.*—(i) Sciama considers the case of a free particle in rectilinear motion in the gravitational field of a mass M which is at rest relative to the smoothed-out universe. If we choose a reference frame at rest relative to the smoothed out universe with this mass M at the space origin, we can neglect in that neighbourhood the deviations from the Galilean values of the $g_{\mu\nu}$ as far as they arise from the universe as a whole, and

include only the deviations due to the mass M. Thus to this approximation the metric will be

$$ds^2 = \left(1 - \frac{2GM}{r}\right)dt^2 - \left(1 + \frac{2GM}{r}\right)(dx^2 + dy^2 + dz^2). \tag{20}$$

It is to be noted that, according to the ideas presented in this paper, the contribution to the $g_{\mu\nu}$ potentials from the universe as a whole is present in the Galilean terms of the $g_{\mu\nu}$.

Since the universe is at rest in this frame we have

$$\left.\begin{aligned} \mathbf{A} &= 0 \\ \text{while} \quad \Phi &= \tfrac{1}{2}\left(1 - \frac{2GM}{r}\right) \end{aligned}\right\}. \tag{21}$$

Suppose the particle is moving freely towards the mass M along the x axis. If its space coordinates are $(x_1, 0, 0)$ at coordinate time t then its coordinate speed is $dx_1/dt = -v$, where $v > 0$. Make now the transformation to a suitable rest frame for the particle, by means of the relations

$$x = X + x_1, y = Y, z = Z, t = T \tag{22}$$

yielding $dx = dX - vdT$, $dy = dY$, $dz = dZ$, $dt = dT$. We get therefore to sufficient order for the covariance of (16)

$$ds^2 = \left(1 - v^2 - \frac{2GM}{r}\right)dT^2 + 2vdXdT - \left(1 + \frac{2GM}{r}\right)(dX^2 + dY^2 + dZ^2). \tag{23}$$

Thus in the particle's rest frame

$$\left.\begin{aligned} \mathbf{A} &= (-v, 0, 0) \\ \Phi &= \tfrac{1}{2}\left(1 - v^2 - \frac{2GM}{r}\right) \end{aligned}\right\}. \tag{24}$$

Apply now equation (17) in the particle's rest frame, yielding

$$-\frac{\partial}{\partial X}\left\{\tfrac{1}{2}\left(1 - v^2 - \frac{2GM}{r}\right)\right\} - \frac{\partial}{\partial T}(-v) = 0 \tag{25}$$

leading to

$$-\frac{GM}{r^2} + \frac{dv}{dt} = 0 \tag{26}$$

on substituting the original coordinates. This is the Newtonian equation of motion of the particle and is also the equation which would follow from the general pondermotive equation (16), applied in the original frame, using (21).

On examining (24) and (25) we see that the origin of the inertial term

$$-\frac{\partial}{\partial T}(-v)$$

in (25) lies in the relative motion of the universe, yielding $\mathbf{A} = (-v, 0, 0)$ in the particle's rest frame, and thus creating an inductive field at the particle which balances the local gravitational attraction due to the mass M, thus connecting with Sciama's ideas.

We note also that the $\mathbf{A}$, Φ in (24) arise by transformation of the *whole* $g_{\mu\nu}$ and not just their deviations from the Galilean values, in accordance with our tentative interpretation of the Galilean values as the static potentials of the whole universe.

(ii) The other case considered by Sciama is that of a particle moving with uniform motion in a circle under the attraction of a mass M at the centre, this mass being again at rest relative to the smoothed-out universe.

Transform therefore from the metric (20) to a suitable rest frame for the particle according to the relations

$$\left.\begin{aligned} x &= X\cos\omega T - Y\sin\omega T \\ y &= Y\cos\omega T + X\sin\omega T \\ z &= Z \\ t &= T \end{aligned}\right\} \tag{27}$$

so that to sufficient order

$$\begin{aligned} ds^2 = (1 - 2GM/R - \omega^2R^2)\, dT^2 - 2\omega\,(-YdXdT + XdYdT) \\ -(1+2GM/R)\,(dX^2+dY^2+dZ^2) \end{aligned} \tag{28}$$

with $R^2 = X^2 + Y^2$.

Thus in this frame

$$\left.\begin{aligned} \mathbf{A} &= (-\omega Y, \omega X, 0) \\ \Phi &= \tfrac{1}{2}\left(1 - \frac{2GM}{R} - \omega^2R^2\right). \end{aligned}\right\} \tag{29}$$

The equation (17) then yields

$$-\frac{GM}{R^2} + \omega^2 R = 0 \tag{30}$$

which is the Newtonian equation of motion, and also, putting $R = r$, what would be given by (16) in the original frame.

Connecting with Sciama's ideas we say that the gravitational attraction by M is balanced by the gravitational field induced by a rotating universe, whose rotational momentum is indicated by $\mathbf{A}$ in (29).

(iii) As a final example we shall show how, by means of the covariance of (16), the Newtonian "fictitious" forces may be attributed to the inductive effect of a moving universe in the most general Newtonian motion of the reference frame relative to a locally inertial frame.

Consider a free particle at rest in a reference frame which is locally inertial, so that the metric is approximately as given by (18) in that region. Let $\mathbf{r}$ be the position vector of the particle in that frame. Then by (15), (16) we have

$$\mathbf{r} = \text{constant}. \tag{31}$$

Transform to a second frame whose space origin has variable velocity $\mathbf{V}$ and which has variable spin $\boldsymbol{\omega}$ relative to the first frame. If the position vector of the particle in this frame is $\mathbf{R}$, then a well-known kinematic result of Newtonian motion gives

$$\dot{\mathbf{r}} = \mathbf{V} + \dot{\mathbf{R}} + \boldsymbol{\omega}\wedge\mathbf{R} \tag{32}$$

$$\ddot{\mathbf{r}} = \dot{\mathbf{V}} + \boldsymbol{\omega}\wedge\mathbf{V} + 2\boldsymbol{\omega}\wedge\dot{\mathbf{R}} + \dot{\boldsymbol{\omega}}\wedge\mathbf{R} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R}) + \ddot{\mathbf{R}} \tag{33}$$

differentiation being with respect to the common Newtonian time of either frame. Thus for the particle in the second frame

$$\ddot{\mathbf{R}} = -[\dot{\mathbf{V}} + \boldsymbol{\omega}\wedge\mathbf{V} + 2\boldsymbol{\omega}\wedge\dot{\mathbf{R}} + \dot{\boldsymbol{\omega}}\wedge\mathbf{R} + \boldsymbol{\omega}\wedge(\boldsymbol{\omega}\wedge\mathbf{R})]. \tag{34}$$

The transformation connecting the two frames is, by (32), in differential form

$$\left.\begin{aligned} d\mathbf{r} &= (\mathbf{V} + \boldsymbol{\omega} \wedge \mathbf{R})dT + d\mathbf{R} \\ dt &= dT \end{aligned}\right\}. \tag{35}$$

Hence $ds^2 = dt^2 - d\mathbf{r}^2$

$$= [1 - \mathbf{V}^2 - 2\mathbf{V}.(\boldsymbol{\omega} \wedge \mathbf{R}) - (\boldsymbol{\omega} \wedge \mathbf{R})^2]dT^2 - 2(\mathbf{V} + \boldsymbol{\omega} \wedge \mathbf{R}).d\mathbf{R}dT - d\mathbf{R}^2 \tag{36}$$

so that in the second frame

$$\left.\begin{aligned} \mathbf{A} &= \mathbf{V} + \boldsymbol{\omega} \wedge \mathbf{R} \\ \Phi &= \tfrac{1}{2}[1 - \mathbf{V}^2 - 2\mathbf{V}.(\boldsymbol{\omega} \wedge \mathbf{R}) - (\boldsymbol{\omega} \wedge \mathbf{R})^2] \end{aligned}\right\}. \tag{37}$$

Now

$$\operatorname{grad} \Phi = \boldsymbol{\omega} \wedge \mathbf{V} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R})$$

$$\frac{\partial \mathbf{A}}{\partial T} = \frac{\partial \mathbf{V}}{\partial T} + \frac{\partial \boldsymbol{\omega}}{\partial T} \wedge \mathbf{R}$$

$$= \dot{\mathbf{V}} + \dot{\boldsymbol{\omega}} \wedge \mathbf{R}$$

while curl $\mathbf{A} = 2\boldsymbol{\omega}$.

Hence by (16) $\ddot{\mathbf{R}} = -[\dot{\mathbf{V}} + \boldsymbol{\omega} \wedge \mathbf{V} + 2\boldsymbol{\omega} \wedge \dot{\mathbf{R}} + \dot{\boldsymbol{\omega}} \wedge \mathbf{R} + \boldsymbol{\omega} \wedge (\boldsymbol{\omega} \wedge \mathbf{R})]$

giving complete agreement with (34).

Thus our theory gives an exact treatment of the fictitious forces as the inductive effect of a moving universe.

6. *Physical interpretation of* **A**, Φ *in general relativity*.—The analysis in this section is intended to be of a tentative nature, since complete rigour cannot be claimed for it.

In Section 4, equation (19), we obtained the result

$$\Phi_0 = \tfrac{1}{2}c^2 = \tfrac{1}{2}g_{44}(0)$$

for that value of the gravitational potential Φ of the whole universe which enters into the pondermotive equation (16), when evaluated at the space origin of a reference frame locally inertial there. In order to interpret this result in terms of Mach's principle we recall the expressions for ϕ in the quasi-Galilean case given by (7) and (11). Since the field equations are relations for the whole $g_{\mu\nu}$ in terms of the matter in the whole universe, we make the tentative inference that in some way the Galilean terms themselves are related to world gravitation, so that inertia would arise in accordance with Mach's principle. To what extent does general relativity provide justification of this inference?

In all cosmological models of general relativity in which the average inertial density ρ does not vanish there is an effective radius R of the model which is the distance, measured in a suitable way, to the horizon of the model where the velocity of the matter relative to the space origin equals the velocity of light. For an observer at the origin matter which goes beyond this distance virtually ceases to exist because of the Doppler effect on its light and presumably on its gravitation. This is the case whether the model be of the homogeneous rotating type (Gödel's models) or the isotropic expanding or contracting types. We shall discuss the latter as an example. These have the general metric

$$ds^2 = c^2dt^2 - e^{g(t)}\frac{(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2)}{(1 + r^2/4R_0^2)^2} \tag{38}$$

where R_0^2 may be positive, negative, or infinite, and the fundamental particles have constant r, θ, ϕ.

The distance to the particle at (r, θ, ϕ), measured in the simultaneity of the fundamental observers at cosmological time t, from $r=0$ is

$$l = e^{\frac{1}{2}g(t)} \int_0^r \frac{dr}{1 + r^2/4R_0^2}.$$

Its radial velocity is therefore

$$\dot{l} = \tfrac{1}{2}\dot{g}l \tag{39}$$

so that

$$|\dot{l}| = c \text{ when } l = \frac{2c}{|\dot{g}|}.$$

Thus

$$R = \pm \frac{2c}{\dot{g}} \quad \text{according as } \dot{g} \gtrless 0 \tag{40}$$

and

$$\dot{l} = \pm \frac{cl}{R}. \tag{41}$$

The importance for Mach's principle is that R is related to σ, the cosmological density of gravitational mass. In general relativity theory, gravitational mass density is defined so as to lead to Gauss' flux theorem for small regions of space (see, for example, Synge (**4**)), and for the isotropic cosmological models $\sigma = \rho + 3p/c^2$ where p is the pressure. The gravitational "force" on unit mass due to the field is in this case the proper acceleration relative to the space origin. With these definitions McCrea (**5**) has shown that the equations of general relativity for the isotropic models are consistent with the variation of the gravitational force according to the Newtonian inverse square law, using proper radial distance but Euclidean geometry, for spatial regions of any size. The field equations applied to the metric (38) give, with $\Lambda = 0$, in general units,

$$\left.\begin{aligned} \frac{8\pi G}{c^2} p &= -\frac{c^2}{R_0^2} e^{-g(t)} - \ddot{g} - \frac{3}{4}\dot{g}^2 \\ 8\pi G \rho &= \frac{3c^2}{R_0^2} e^{-g(t)} + \frac{3}{4}\dot{g}^2 \end{aligned}\right\} \tag{42}$$

so that

$$8\pi G\sigma = -3(\ddot{g} + \tfrac{1}{2}\dot{g}^2).$$

Now for $\dot{g} > 0$, $\quad \dot{g} = \dfrac{2c}{R}, \; \ddot{g} = -2c\dot{R}/R^2$ by (40). Hence

$$8\pi G\sigma = -\frac{6c^2}{R^2}(1 - \dot{R}/c)$$

$$\left.\begin{aligned} &\text{whence, if } \dot{g} > 0, && G\sigma R^2 = -\frac{3c^2}{4\pi}(1 - \dot{R}/c) \\ &\text{and, if } \dot{g} < 0, && G\sigma R^2 = -\frac{3c^2}{4\pi}(1 + \dot{R}/c) \end{aligned}\right\} \tag{43}$$

which are the required relations between σ and R, at time t. For $\dot{g} > 0$ we see that $\sigma \gtrless 0$ according as $\dot{R} \gtrless c$. If $\dot{R} > c$ matter is entering the region bounded by the defined horizon; if $\dot{R} < c$ matter is passing beyond this horizon. For $\dot{g} < 0$, $\sigma \lessgtr 0$ according as $\dot{R} \lesseqgtr c$.

By (41) and (43) we get the Newtonian type equation

$$\ddot{l} = -\tfrac{4}{3}\pi G\sigma l \tag{44}$$

relating gravitational force and proper distance at time t.

Comparing equation (44) with the pondermotive equation (16) we see that, for a fundamental observer whose radial space coordinate is the proper distance l and whose time is the cosmological time t, $\mathbf{A}=0$ and $-\text{grad}\,\Phi = -\frac{4\pi}{3}\pi G\sigma\mathbf{l}$. Equation (44) however holds for all $l \leqslant R$ and not just in the neighbourhood of the origin. Consider therefore the gravitational "work" done by the field when a particle of unit mass is moved from its actual position at time t to the horizon and therefore beyond influence of the origin. This will be

$$\Phi_l = -\frac{4}{3}\pi G\int_l^R \sigma l dl. \tag{45}$$

This may be regarded as the analogue of the Newtonian potential at the distance l, in the gravitational field as witnessed by an observer at the origin. Both σ and R will vary with l in this integral as the motion proceeds, according to (42), (43). However Φ_l may be evaluated as

$$\begin{aligned}\Phi_l = \int_l^R \ddot{l}dl &= \tfrac{1}{2}c^2 - \tfrac{1}{2}\dot{l}^2\\ &= \frac{c^2}{2}(1 - l^2/R^2).\end{aligned} \tag{46}$$

This is the potential at $\mathbf{l}$ at time t.

Putting $l=0$ we get
$$\Phi_0 = c^2/2 \tag{47}$$
which therefore provides a physical identification, in a natural way, of the potential Φ_0 arising in equation (19).

The result given by (47) for Φ_0 is got as the limit of Φ_l when l tends to zero irrespectively of the sign of σ or $\dot{g}$. For instance if $\sigma > 0$ and $\dot{g} > 0$ then $\dot{R} > c$, so that for the field to carry the particle to the horizon of the space origin would mean going backwards in time. The discrepancy in sign between our Φ_0 and Sciama's, referred to at the end of Section 4, arises because of Sciama's arbitrary definition of Φ for an expanding universe. His definition appears to ignore the above considerations and in particular to presuppose the identity of the cosmological gravitational mass density and the inertial density.

For cogent reasons which have been put forward elsewhere (**6**) a stationary cosmological solution is to be preferred. The only known stationary solution which does not contradict observational results (expansion, spatial isotropy) is the steady state theory proposed by Bondi and Gold (**6**). This has the metric

$$ds^2 = c^2dt^2 - e^{2ct/R}(dr^2 + r^2d\theta^2 + r^2\sin^2\theta d\phi^2) \tag{48}$$

where R is a constant which is the effective radius for the model. The steady state model, unlike the general cosmological models of metric given by (38) for which equation (16) vanishes identically, allows a static metric to be used so that the motion of a particle, relative to the observer at the origin, is measured by the rate of change of the spatial coordinates. This is the De Sitter metric

$$ds^2 = c^2(1 - l^2/R^2)d\tau^2 - \frac{dl^2}{1 - l^2/R^2} - l^2d\theta^2 - l^2\sin^2\theta d\phi^2 \tag{49}$$

connected to (48) by a well known transformation. In this metric l is the distance from the origin, in the simultaneity of the fundamental observers, of our general analysis. The theory of Section 4 defines the Φ involved in (16) as $\frac{1}{2}g_{44}$ which in the case of the metric (49) gives

$$\Phi = \frac{c^2}{2}(1 - l^2/R^2) \tag{50}$$

agreeing with (46) and therefore having the physical interpretation associated with (46).

It is to be noted that the steady state forms a natural cosmological background to mass concentrations. For instance the *exact* solution of the field equations for an isolated mass m superimposed on the steady state is

$$ds^2 = c^2\left(1 - \frac{2mG}{c^2 l} - \frac{l^2}{R^2}\right)d\tau^2 - \frac{dl^2}{1 - \frac{2mG}{c^2 l} - \frac{l^2}{R^2}} - l^2 d\theta^2 - l^2 \sin^2\theta d\phi^2. \tag{51}$$

For this metric

$$\Phi = \frac{c^2}{2}(1 - l^2/R^2) - \frac{mG}{l} \tag{52}$$

to be interpreted physically as the work done by the field in removing unit mass from the point in question to the horizon of the model, regarding mG/R as negligible.

A Newtonian type integral for Φ in terms of the distribution of the mass does not follow simply in the case of the general models because of the stated dependence of σ and R on cosmological epoch. However, for the steady state, σ and R are constant and we may write for the potential of unit mass, at distance l from the mass σdV constantly in the volume element dV,

$$\begin{aligned} d\Phi &= -G\int_l^R \frac{\sigma dV}{l^2}\, dl \\ &= -G\sigma dV\left(\frac{1}{l} - \frac{1}{R}\right). \end{aligned} \tag{53}$$

Thus

$$\begin{aligned} \Phi_0 &= -G\sigma\int_0^R \left(\frac{1}{l} - \frac{1}{R}\right) 4\pi l^2 dl \\ &= -\frac{2\pi}{3} G\sigma R^2. \end{aligned}$$

Equation (43) gives for the steady state $G\sigma R^2 = -3c^2/4\pi$ so that

$$\Phi_0 = c^2/2$$

in agreement with (47) and justifying our physical interpretation of Φ_0 as the gravitational potential of all the matter in the universe apparent to an observer at the origin and having influence there.

The quantity **A** of our theory defined in equation (15) of Section 4 is zero for the cosmological metric (38). On making a transformation such as that of the first example in Section 5, equation (22), a non-zero **A** arises by transformation of the $g_{\mu\nu}$. If we accept the association of the Galilean values of the $g_{\mu\nu}$ with world gravitation, according to the tentative analysis presented above, the association of **A** with the relative momentum of the universe would also follow. While the Galilean g_{44}, viz. c^2, is associated with Φ_0 as $2\Phi_0$, the spatial Galilean $g_{\mu\nu}$ are associated with **A**. Thus in the first example in Section 5 to get **A** in the particle's rest frame we have to multiply the Galilean g_{11} in the original frame, viz. $-1 = g_{11}(0)$, by v. According to the subsequent application of the pondermotive equation $-g_{11}(0)$ is proportional to the inertial mass of the particle, the whole equation indicating equality of gravitational and inertial mass in the case of a particle. Since $-g_{11}(0) = g_{44}(0)/c^2 = 2\Phi_0/c^2$ we see that inertial mass can therefore be associated with the influence of the whole universe, in accordance with Mach's principle.

7. *Comparison with Sciama's theory.*—In this final section we shall remark briefly on the three principal differences claimed by Sciama between his theory and general relativity, enumerated (i), (ii), (iii) in the introduction to this paper.

(i) It is evident from the analysis in this paper that a knowledge of G, occurring in the integral (45) leading to (47), together with R given as cT where T is the reciprocal of the Hubble parameter, leads to an estimate of the amount of matter in the universe in general relativity as well as in Sciama's theory.

(ii) Sciama states that in general relativity the principle of equivalence predicts that one gravitating mass in an otherwise empty universe produces the same inertial effects as in his theory, and since there is no universe in this case to give rise to the inductive field "it is difficult to see why the principle of equivalence should be true". Such an argument however implies a solution of the field equations involving the use of coordinates for all points of space-time in a universe which, except for the isolated mass, is empty. Such coordinates are purely conceptual, defined without reference to matter and restoring to space an objective substance, independent of matter, which general relativity has sought to deny. The logical course for general relativity, according to the field equations, is to relate the Galilean $g_{\mu\nu}$ of special relativity to world gravitation in a full universe. That general relativity may be capable of doing so has been indicated in this paper, where, independently of the value of ρ as long as it does not vanish, $\frac{1}{2} g_{44}(0)$ has been identified as $\Phi_0 = c^2/2$, the potential of the universe. The case of the empty universe can only logically be approached as a limit where $\rho \to 0$ and $R \to \infty$ (equation (43)), so that inertia is always accounted for.

(iii) Sciama's determination of the sign of the field in his theory is of doubtful significance as it depends on his Φ as defined turning out to be negative. Reasons for questioning this arbitrary definition of Φ have been given in Section 6.

In general relativity the coefficient of $T^{\mu\nu}$ in the field equations is chosen so as to make gravitation attractive on the small scale (the pressure being then relatively negligible). However, on the cosmological scale this leads to gravitational mass being interpreted as negative if the expansion is accelerating (equation (44)). Thus there would appear to be no intrinsic importance to be attached to the sign of the field in any theory that allows for factors at present unknown, that is whether the gravitational effect of a lump of matter is more primary than that of a cosmological region containing "zero-point" stress.

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SURVEY OF BRIGHT GALAXIES SOUTH OF $-35°$ DECLINATION*

G. de Vaucouleurs

Summary

A survey of the bright galaxies listed in the Shapley–Ames Catalogue (*H.A.*, **88**, 2, 1932) south of declination $-35°$ was carried out between 1952 and 1955 with the 30-inch Reynolds reflector of the Commonwealth Observatory, Mount Stromlo. About 250 one-hour exposures were obtained on Eastman 103a–o emulsion with the reflector diaphragmed to 20 inches ($f/6$, scale $67''$/mm). New or revised types and dimensions were determined for 210 S–A objects (out of 230 listed) and in addition for 120 other NGC and IC objects and also for 130 uncatalogued objects, a total of 460 galaxies generally larger than $1'$ and/or brighter than the 14th magnitude.

Morphological types were assigned in a revised classification scheme incorporating the improved Mt Wilson–Palomar system of Hubble and Sandage, as modified and supplemented by the present Mt Stromlo work. It includes four main *classes*: ellipticals (E), lenticulars (SO), spirals (S) and irregulars (I); two *families* of lenticulars and spirals: ordinary (SA) and barred (SB), with intermediate objects (SAB); two main *varieties* of lenticulars and spirals: ringed (r) and spiral (s), with intermediate objects (rs); four or five *sub-divisions* or stages along the spiral sequences: early (a), intermediate (b), late (c, d) and magellanic (m), the latter forming the transition toward the magellanic irregulars I (m). Three stages, early, intermediate and late are also distinguished in the lenticular sequences (SO^-, SO, SO^+) between ellipticals and spirals. Outer ring-like structures (R) are frequently observed near the transition stage SO/a between lenticulars and spirals (Section 3 and Plate I). The apparent *frequency of types* among the S–A objects in the zone is: E(23·4 per cent), SO (21·0 per cent), SA (24·4 per cent), SB (26·3 per cent), I (4·9 per cent), (Section 6). The possible evolutionary significance of the classification sequence is briefly discussed (Section 16).

Dimensions obtained by visual measurement of the negatives and reduced to standard conditions are given for the bright, inner regions ($D_i x d_i$) and for the faint, outer regions ($D_0 x d_0$), (Section 4). The median outer diameters D_0 for the S–A objects in the zone are: E ($1'4$), SO ($2'0$), S ($3'2$), (Section 7); there is no perceptible latitude effect (Section 10). However, apparent diameters depend on apparent flattening, i.e. tilt, and a correction, independent of type, must be applied (Section 9). The ratio of inner to outer dimensions (D_i/D_0), measuring the *concentration*, depends also on flattening (Section 8); the variations of corrected diameter and of concentration index *vs.* galaxy type are determined (Section 11).

The *axial ratio* (d/D) of ellipticals is discussed (Section 12); there is no excess of either globular or elongated shapes compared with a random distribution of true oblateness and tilt. On the contrary, lenticulars (Section 13) show a large excess of strongly elongated shapes as expected; but spirals (Section 15) of corrected diameters larger than $3'$ show a deficiency of elongated shapes which may indicate some restriction on the spatial orientation of the nearer galaxies.

* The full text of this paper appears in *Memoirs of the Commonwealth Observatory, Mount Stromlo*, **3**, No. 13.

Ring-structures in spirals are of special interest as possible geometric distance indicators. A preliminary discussion (Section 17) of 35 objects indicates that the mean diameter of the inner ring (r) is constant throughout the spiral sequences from SO/a to Scd; the range among objects of known velocities appears to be less than 2 to 1 with a mean value of about 2·5 kiloparsecs and a variance of about $\pm$ 15 per cent. Ring diameters appear potentially capable to serve as useful distance indicators out to about 100 megaparsecs. The fainter outer rings (R) are about 1·7 times larger on the average but are often too faint to be detected in small objects.

Relative distances based on apparent diameters are determined for seven clouds or regions outlined by the apparent distribution of the S–A objects in the zone (Section 5 and Plate II). An excellent agreement is observed between the various determinations based on ellipticals, lenticulars, spirals and rings. Absolute distances based on ring diameters are derived (Section 18) which vary from about 7 megaparsecs for a group of spirals in Grus (region VII) to 10 megaparsecs for the Pavo–Indus cloud (region VI). A dozen foreground objects of outstanding apparent diameters are at distances of from 1 to 3 megaparsecs.

Relative diameters at unit distance are determined as a function of type along the classification sequence (Section 19). The range of mean diameters from late spirals to ellipticals is only 2 to 1 in contrast with early Mt Wilson results, but in good agreement with Harvard data on the Virgo cluster. The frequency functions of diameters give for most types some indication of a bimodal distribution with a small group of supergiant and a main group of giant galaxies; dwarf systems are practically excluded by the magnitude–surface brightness selection of the S–A Catalogue.

Pairs and triplets are classified according to the visibility or otherwise of tidal distortion in the components (Section 20). For 30 pairs showing no clear sign of interaction the mean projected separation is 18 kiloparsecs (range 8 to 45 kpc); for 8 pairs showing distortion effects it is 6·5 kiloparsecs (range 3 to 8 kpc). The mean critical separation within which tidal distortion becomes notable is of the order of 9 kiloparsecs (actual). *Two colliding systems* (NGC 1487, NGC 3256) are described; neither of them is a radio source. Some dense *groups and clusters* were noted in Fornax, Antlia, Centaurus, Telescopium and Pavo (Section 21). The densest, around NGC 4696, appears to be associated with a radio source.

Comparisons with earlier surveys including objects south of -35° indicate notable under-exposure for the older Helwan and Harvard (*H.A.*, **88**, 2) series; there is good agreement, however, with the more recent Harvard series (*H.A.*, **88**, 4, 1934), especially for spirals; the mean scale factor is HA 88,4/Stromlo (D_0) = 1·2 and the average deviation $\pm$ 8 per cent. Reduction factors to the scale of the Heidelberg survey of northern nebulae (Reinmuth's Catalogue) are obtained via *H.A.*, **88**, 4.

Four lists give general information and details of the Reynolds reflector data on (I) S–A objects south of -35° (main catalogue), (II) additional NGC and IC objects, (III) large uncatalogued objects, (IV) pairs of galaxies, in the same zone.

Eight plates give (I) a description of the classification system, (II) a map of the apparent distribution of the S–A objects south of -35°, (III to VI) Reynolds reflector photographs of 12 large spirals, and (VII, VIII) photographs of 32 other bright galaxies illustrating the classification system and some interesting objects.

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9. Postcards giving the programme of each meeting are issued some days before the meeting concerned. Fellows wishing to receive such cards whether for Ordinary Meetings or for the Geophysical Discussions or both should notify the Assistant Secretary.

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